Problem 1 Everything in the world is at least two of

1. i. A dolphin
2. ii. A mathematician
3. iii. A tiger

A dolphin can not also be a tiger. Each group is very direct, it is known that either their entire group hates pianos, or their entire group likes pianos. If all dolphins hate pianos, is it necessarily true that all tigers also hate pianos?

Solution Note that all dolphins must be mathematicians, as they cannot be a tiger, thus all mathematicians must hate pianos. Similarly all tigers must be tiger-mathematicians, since all mathematicians hate pianos, all tigers must hate pianos as well. Hence, the statement is $\boxed{\text{True}}$.

Problem 2 Evaluate

$$(-100^{-100})(-99^{-99})...(99^{99})(100^{100})$$

Solution In the middle of the expression, we find 0^0, which means the product is undefined and has $\boxed{\text{No Value}}$. $\square$

Problem 3 In a game Ralph's last game of ballbasket, he scored a total of 30 points. His score was a combination of shots that were worth 2 point and 3 points. If Ralph did **not** score more 3 pointers than 2 pointers, how many 2 pointers and 3 pointers did he score?

Solution Let the number of 2-pointers and 3-pointers Ralph scores be a and b respectively. We have

$$2a + 3b = 30$$

If $a = b$, then we have $a = 6$ which is our first solution. From this, we know $6 < a$. We can also see that $2a = 30 - 3b = 3(10 - b)$, which means a must be divisible by 3. Thus, there are three solutions, Namely $\{a, b\} = \boxed{\{6, 6\}, \{9, 4\}, \{12, 2\}}$.

Note that any of the above answers will net you full points.

Problem 4 There are 10 cards in the deck, 8 red and 2 blue. Ralph and Oscar are dealt 5 cards at random. They each pick a random three cards from their cards and place it onto the board. What is the probability that the 6 cards placed on the board are all red?

Solution Note that first picking the final 6 cards that end up on the table then distributing the cards to each of the players is the exact same as dealing it out the other way around. Thus the final probability is how often we can pick 6 red cards from 8 red and 2 blue cards.

$$\frac{\binom{8}{6}}{\binom{10}{6}} = \boxed{\frac{2}{15}}$$

Problem 5 A beam of light is shot at a 45 degree angle from a flat surface. Two mirrors are set up parallel to each other and perpendicular to the surface such that the beam of light - in order - passes under the first mirror, hits the second and then goes above the first mirror again. If the mirrors are 3 metres apart, what is the maximum length of the first mirror?

Solution The largest possible length of the mirror will touch, but not go through the beam of light. Let A and B be the endpoints of the first mirror, and C be the point the beam of light hits the second mirror. $\angle ACB = 2 \times 45^\circ = 90^\circ$ By symmetry, $\triangle ABC$ is isosceles, and thus a 45-45-90 triangle. We know the altitude of $\triangle ABC$ from C is 3 metres, so $AB = \boxed{6m}$

Problem 6 Joseph and Ron run on a circular track of length 5 km. It is known Joseph and Ron run clockwise at a constant speed of 6 km/h and 9 km/h respectively and start at the same point. Let the next point Joseph and Ron meet be n km clockwise from their starting point. Find n.

Solution Since Ron runs faster than Joseph, the next point they meet is when Ron laps Joseph. This means that Ron will have run an extra 5 km compared to Joseph. Let the time it took to run be t

$$\begin{aligned} rt &= d \\ 6(t) &= d \\ 9(t) &= d + 5 \\ (9-6)t &= 5 \\ t &= \frac{5}{3} \end{aligned}$$

Solving for d gives us $d = 6\left(\frac{5}{3}\right) = \boxed{10km}$. This point can also be seen as $\boxed{0km}$ from the starting point.

Problem 7 A pandigital number is a positive integer such that all digits are included (not necessarily only once). For example, 442018935867 is a pandigital number in base-10, as it uses all digits from 0-9. Find the smallest integer such that it is pandigital when expressed in base-3 and in base-4. Write your answer in base-10.

Addendum: A base-b number $\overline{a_1 a_2 ... a_n}$ can be expressed as $a_1 b^{n-1} + a_2 b^{n-2} + ... + a_n b^0$

Solution The smallest pandigital number in base-4 is 1023_4, we note $75 = 1023_4 = 22010_3$. Which is also pandigital. Thus $\boxed{75}$ is the smallest number that is pandigital in both base-3 and base-4. $\square$

Problem 8 We define the new function Ξ, such that

$$x \;\Xi\; y = \begin{cases} x \text{ if } x \text{ and } y \text{ are coprime} \\ x + y \text{ if } x \text{ and } y \text{ are not coprime.} \end{cases}$$

Two number are coprime if they have no common factors other than 1.

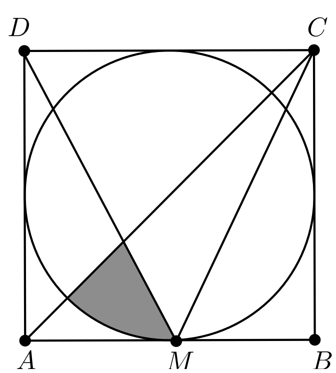
Evaluate 2 Ξ (3 Ξ (...(99 Ξ 100)...)

Solution We note that any two consecutive numbers MUST be coprime. This can be proved by contradiction. If $gcd(n, n+1) = d \neq 1$ for some positive integers n, d then, for some integer k, we can write this as

$$dk_1 = n, \quad dk_2 = n + 1$$
$$d(k_1 - k_2) = 1$$
$$k_1 - k_2 = d = 1$$

Which is a contradiction. Thus only the first case of the new function will occur, and our final answer will be $\boxed{2}$

Problem 9 Let ABCD be a square with point M be the midpoint of side AB. Inscribe a circle of center O in the square. Let N be the intersection of lines AC and MD. Find the ratio of the area of the shaded region to the area of the square.



Solution Let P be the intersection between AC and DM. Let N be the midpoint of AD. To solve, we note

$$[\text{Shaded Region}] = [ADM] - \frac{[AMN]}{2} - [ADP]$$

Where [AMN] is bounded by the small arc MN. If the side length of the square is 1, we have

$$[ADM] = \frac{(1)(\frac{1}{2})}{2} = \frac{1}{4}$$

$$\frac{[AMN]}{2} = \frac{1 - \pi(\frac{1}{2})^2}{(4)(2)} = \frac{4-\pi}{32}$$

Where we found $[AMN]/2$ by subtracting the circle's area from the square's area and dividing it by 8. To find [ADP] we introduce a coordinate system. Place our diagram on the Cartesian plane such that A (0, 0), D (0, 1), and M (0.5, 0). Using point-slope form, we find the slope of DM is -2, and the slope of AC is 1. We build our system of equations to solve for N. We are looking for the distance from N to the line AD (the altitude) so we only need to solve for x.

$$\begin{aligned} 1(x-0) &= y-0 \\ -2(x-0) &= y-1 \\ -2(x-0) &= x-1 \\ x &= \frac{1}{3} \end{aligned}$$

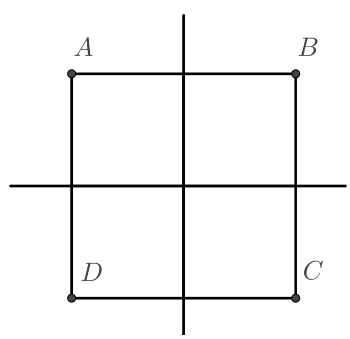
Thus,

$$[ADM] = \frac{1(\frac{1}{3})}{2} = \frac{1}{6}$$

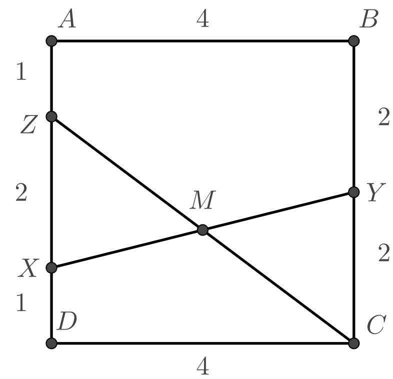
$$\frac{\text{Shaded Region}}{1} = \frac{1}{4} - \frac{4-\pi}{32} - \frac{1}{6} = \boxed{\frac{3\pi-4}{96}}$$

Problem 10 Cut a square with side length 4 into four distinct non-overlapping parts with two straight lines such that the area of the four parts make the ratio 1:1:2:4. Label the length of all new sides of the square.

Example of using two lines to cut the square into a 1:1:1:1 ratio



Solution The total area is 16, so to obtain the desired ratio, we must cut the square into pieces of are 2, 2, 4, and 8. We cut up the square as follows. Draw the first line from C to a point Z on AD such that AZ=1. Let the second line intersect AD and BC and X and Y respectively such that AX=3 and BY=2. Let CZ and XY intersect at M



We know $AD \parallel BC$, so $\angle DZC = \angle ZCY$ and $\angle MXZ = \angle MYC$. We performed the construction such that $XZ = CY = 2$, thus $\triangle MZX \cong \triangle MCY$. This means that M is the mid point of XY, thus the altitude of both $\triangle MZY, \quad \triangle MCY$ is $\frac{4}{2} = 2$. We can now solve for the areas of our pieces.

$$[MZX] = [MCY] = \frac{2 \cdot 2}{2} = 2$$
$$[MCDX] = [CDZ] - [MZX] = (\frac{4 \cdot 3}{2}) - (2) = 4$$
$$[MZABY] = [ABCD] - ([CDZ] + [MCY]) = 16 - (6 + 2) = 8$$

Thus $[MZX] : [MCY] : [MCDX] : [MZABY]$ follows the desired 1:1:2:4 ratio . $\square$

Problem 1 Given that x, y are positive integers and $x \neq -y, -2y, -3y, -4y$. Find which expression always outputs a larger value

$$\frac{1}{\frac{4}{x+3y} - \frac{4}{x+4y}} - \frac{1}{\frac{4}{x+y} - \frac{4}{x+2y}} \qquad or \qquad x + 2y$$

Solution We note that the restriction on the bottom prevents dividing by zero. Multiplying out the denominators and simplifying the right expression gives

$$\frac{1}{\frac{4}{x+3y}-\frac{4}{x+4y}}-\frac{1}{\frac{4}{x+y}-\frac{4}{x+2y}}=\frac{1}{\frac{4(x+4y)-4(x+3y)}{(x+3y)(x+4y)}}-\frac{1}{\frac{4(x+2y)-4(x+y)}{(x+y)(x+2y)}}$$

$$=\frac{1}{\frac{4y}{(x+3y)(x+4y)}}-\frac{1}{\frac{4y}{(x+y)(x+2y)}}$$

$$=\frac{(x+3y)(x+4y)-(x+y)(x+2y)}{4y}$$

$$=\frac{4xy+10y^2}{4y}$$

$$=x+\frac{5}{2}y$$

x, y are both positive integers, so we know $x+\frac{5}{2}y > x+2y$. Thus, $\boxed{\frac{1}{\frac{4}{x+3y}-\frac{4}{x+4y}}-\frac{1}{\frac{4}{x+y}-\frac{4}{x+2y}}}$ is always greater. □

Problem 2 Jack wants to line up 7 books on a bookshelf cover to cover The width of the books are 1 unit wide, if the length of the books are $2, 4, 8, ..., 2^7$. Perimeter made up of the spines of the books. Out of all possible arrangements what is the largest perimeter Jack can create?

Solution We can see that the total perimeter will be

$$\text{total perimeter } -\sum \text{overlaps}$$

The total perimeter can be found using a geometric series, or just by addition,

$$(2\times 7)+2(2^1+2^2+...+2^7)=14+2\left(\frac{2^8-1}{2-1}\right)=522 \qquad \text{by geometric series}$$

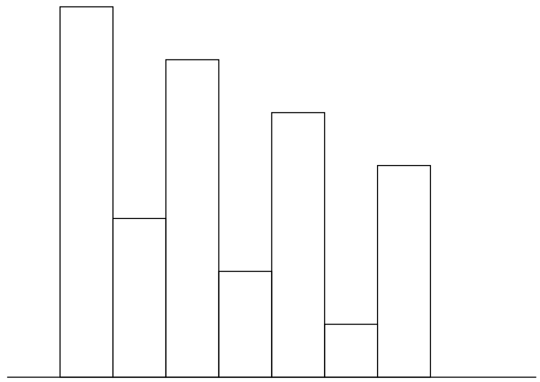
$$=522 \qquad \text{or by adding}$$

Thus, we want to minimize the overlap. The overlap between any two books is determined solely by the smaller of the two books, and there will be 6 places overlap occurs. Since any book will contribute to an overlap twice (one on each side) we will space out the books such that only the smallest books effect the overlaps. The smallest three books are of length 2, 4, and 8, so

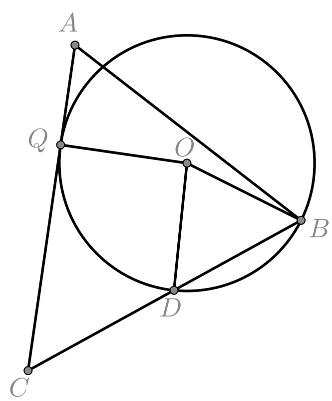
$$\sum \text{overlap} = 4(2+4+8)=56$$

$$522 - 56 = \boxed{466}$$

We can see this is possible in the below diagram (numbers are scaled down to make my life easier)



Problem 3 Given ΔABC with D on BC. Let the circle with centre O be tangent to AC at Q and pass through both B and D. Prove that $\angle ACB = \angle QDO$ if and only if O is on line BQ.



Solution Assume $\angle ACB = \angle QDO$. Consider the angles in quadrilateral BCQO

$$360^\circ = \angle OQC + \angle ACB + \angle DBO + \angle BOQ$$
$$360^\circ = (90^\circ) + \angle ACB + (360^\circ - \angle QDO - \angle QDB)$$

Using OQDB as a quadrilateral

$$\begin{aligned} 90^\circ + \angle ACB &= \angle QDO + \angle QDB \\ &= \angle QDO + (\angle QDO + \angle ODB) \\ &= \angle ODB + 2\angle QDO \end{aligned} \tag{1}$$

Since $\angle ACB = \angle QDO, \angle ODB + \angle QDO = \angle QDB = 90°$. The hypotenuse of a circumscribed triangle always passes through the centre of the circle, hence O is on QB.

Conversely, assuming O is on QB, we will prove $\angle QDO = \angle ACB$. We know $\triangle BDQ$ is a right triangle, and both $\triangle QOD$ and $\triangle BOD$ are isosceles, thus, by (1) (note that this was proved before using any assumptions), we have

$$\begin{aligned}(\angle ODB + \angle QDO) + \angle QDO &= 90° + \angle ACB\\ (90°) + \angle QDO &= 90° + \angle ACB\\ \angle QDO &= \angle ACB \qquad \square\end{aligned}$$

Problem 4 A 4x4 board can be filled with M, A, T, and H. Chuck the Chomper enjoys chomping on M, A, T, and H's. However, Chomp will only binge chomp, so he will only chomp on M, A, T, and H's when there is both an entire row and column of M, A, T, and H's. If a square has already been chomped on, Chuck won't acknowledge its existence. If the board is initially filled out as done in the diagram below, how many different ways can you finish tiling the 4×4 board such that all M, A, T, and H's will be chomped on by the end?

H			
T			
A			
M	A	T	H

Solution Note that we can create a unique solution by finding how many ways to pick the number of rows or columns. This can be done in $6! = 720$ ways.

Note: Due to being incredibly confusing, this question was omitted.

Problem 5 A unidigital number is one that is constructed out of only one digit. Find the number of 2's in a unidigital number made exclusively out of 2's and is divisible by 2018. (Note: There are several solutions, but only one final answer) (Source: Canada)

Solution I claim the answer is $\boxed{1008}$. To prove this is divisible by 2018, first note that since it is made exclusively out of 2's, our answer will be divisible by at least 2. Hence, we will prove that our number is divisible by 1009, which is a prime. We can write 1008 twos as the sum of powers of ten.

$$2(10^{1007} + 10^{1006} + ... + 10^1 + 10^0) = 2\left(\frac{10^{1008} - 1}{9}\right)$$

By Fermat's Little Theorem the $a^{1008} - 1 \equiv \pmod{1009}$, for any a not divisible by 1009 which is what we wanted. Since 2018 is not divisible by three, the problem is solved.

Problem 6 Ralph only reads prime numbered pages. Oscar only reads composite numbered pages. Starting from 1, what is the maximum number of pages in the book such that the number of pages Oscar reads is exactly double the number of pages Ralph reads?

Solution Since 1 is neither prime nor composite, we start counting the ratio of primes to composites from 2. The list of primes {2,3,5,7,11,13,17,19,23,29,31,37,41,43} contains exactly 14 elements. Thus we look for the 28th composite number, which is $\boxed{43}$. To prove this is the maximum, note that every 6 consecutive numbers there will be the residues $0, 2, 3, 4 \pmod 6$ which are always divisible 2 or 3, and thus at least 4 of them will be composite. Hence, from 43, the ratio between the number of pages Ralph reads and Oscar reads can never decrease between any numbers that differ by 6. It is easy to see the next 6 numbers all have a > 2 composite to prime page ratio. $\square$