2022 Team Math Attack Contest

Team Contest Solution Set

December 17, 2022

Team Contest Solutions

1. A palindrome is a whole number whose digits read the same backwards as forwards. What is the smallest 5-digit palindrome?

   Answer: $\boxed{10001}$

   We know this number has to be greater than 10000 due to having 5 digits. To minimize the number, we set the first digit as 1, making the unit digit 1 as well. We can then let everything else be 0.

2. How many positive odd integers are a factor of 2022?

   Answer: $\boxed{4}$

   2 x 3 x 337 results in 1, 3, 337, 1011 as factors.

3. During a road trip, Bill drove 580 kilometres using 16 litres of gas. His car drives 40 kilometres per litre on highways and 25 kilometres per litre in cities. How many kilometres of his road trip were on the highway?

   Answer: $\boxed{480}$

   Two equations: 40n + 25x = 580 and n + x = 16, where n is the litres of gas used on the highway and x is the litres of gas used in cities. Substitute x for 16-n, isolate, and n = 12 litres. 12 L * 40 km/L equals to 480 km.

4. How many triangles have a perimeter less than 15, one side with a length of 3, and the other two sides with prime number side lengths?

   Answer: $\boxed{5}$

   We use casework to count each solution, and find 5 solutions, (2,3,3),(2,2,3),(3,3,5),(3,5,5),(3,3,3)

5. How many **different** arrangements of the letters "MATHATTACK" are possible? Don't forget, MATH-ATTACK counts as a arrangement too!

Answer: $\boxed{100800}$

10!/3!/3!=100800

6. In a deck of 52 cards, if 13 of them are hearts, what is the probability that a randomly-dealt hand of 5 cards will contain at least one heart?

   Answer: $\boxed{7411/9520}$

   1 - (39C5/52C5) = 7411/9520

7. A triangle lies on a plane and has the coordinates $(4, 3)$, $(10, 11)$, and $(10, 8)$. What is the perimeter of the triangle?

   Answer: $\boxed{13 + \sqrt{61}}$

   To find the distance between $(4, 3)$ and $(10, 11)$ we can use the Pythagorean theorem to get $\sqrt{(10-4)^2 + (11-3)^2} = \sqrt{36 + 64} = 10$. Similarly, we can find the other distances to be 3 between $(10, 11)$ and $(10, 8)$ and $\sqrt{(10-4)^2 + (8-3)^2} = \sqrt{36 + 25} = \sqrt{61}$ between $(4, 3)$ and $(10, 8)$ to get $3 + 10 + \sqrt{61} = 13 + \sqrt{61}$.

8. What is the coefficient of the a^8d^2 term of the expansion of $(2a + 5d)^{10}$?

   Answer: $\boxed{288000}$

   Using the binomial theorem, we get a term of 288 000 a^8d^2. Thus, the coefficient is 288 000.

9. A chip bag has 2 ketchup chips, 4 all-dressed chips, and 7 dill pickle chips. If you eat two chips, what is the probability of eating two different chips?

   Answer: $\boxed{25/39}$

   Probability of eating 2 ketchup $= \frac{2}{13}\frac{1}{12} = \frac{2}{156}$

   Probability of eating 2 all-dressed $= \frac{4}{13}\frac{3}{12} = \frac{12}{156}$

   Probability of eating 2 pickle $= \frac{7}{13}\frac{6}{12} = \frac{42}{156}$

   Probability of not eating 2 of same chip $= 1 - \frac{2}{156} - \frac{12}{156} - \frac{42}{156} = \frac{156-56}{156} = \frac{100}{156} = \frac{25}{39}$

10. How many four digit numbers have exactly 9 divisors from the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$?

    Answer: $\boxed{3}$

    Looking through the numbers, you can see that 1, 2, 3, 4, 5, 6, and 10 must all be used. Therefore, from 7, 8, and 9 we must pick 2 to be used and the third to be left out. When 7, 8, and 9 are left out, there are 25, 7, and 10 solutions respectively. There are also three overlapping solutions between all of them, so the final answer is 33 solutions.

11. There are 500 cups in a circle, and a ball is placed in every 11th cup starting with the 1st cup. Balls are placed in cups until another ball is placed in the 1st cup. Which cup is the 271st ball placed in?

Answer: $\boxed{471}$

Each ball is placed in 11 further cups from the first one. Therefore, the 271st ball will be placed in 270*11 = 2970 further cups from the first cup. Every 500 cups, it will return back to the first cup. After 2500, there are an additional 470 cups from cup 1, so it will be put in the 471st cup.

12. If $\frac{1}{x} + x = 5$, then what is the value of $\frac{1}{\sqrt{x}} + \sqrt{x}$?

    Answer: $\boxed{\sqrt{7}}$

    Set the unknown expression equal to a variable, square it, plug in the first equation and solve.

13. A box contains a collection of triangular and square tiles. There are 25 tiles in the box, containing 84 edges in total. How many triangle tiles are there in the box?

    Answer: $\boxed{16}$

    Let t be the number of triangle tiles. Then there are $3t + 4(25 - t)$ edges $= 84$ edges. We solve this equation to get $t = 16$.

14. What is the sum of the digits of 999999^2?

    Answer: $\boxed{54}$

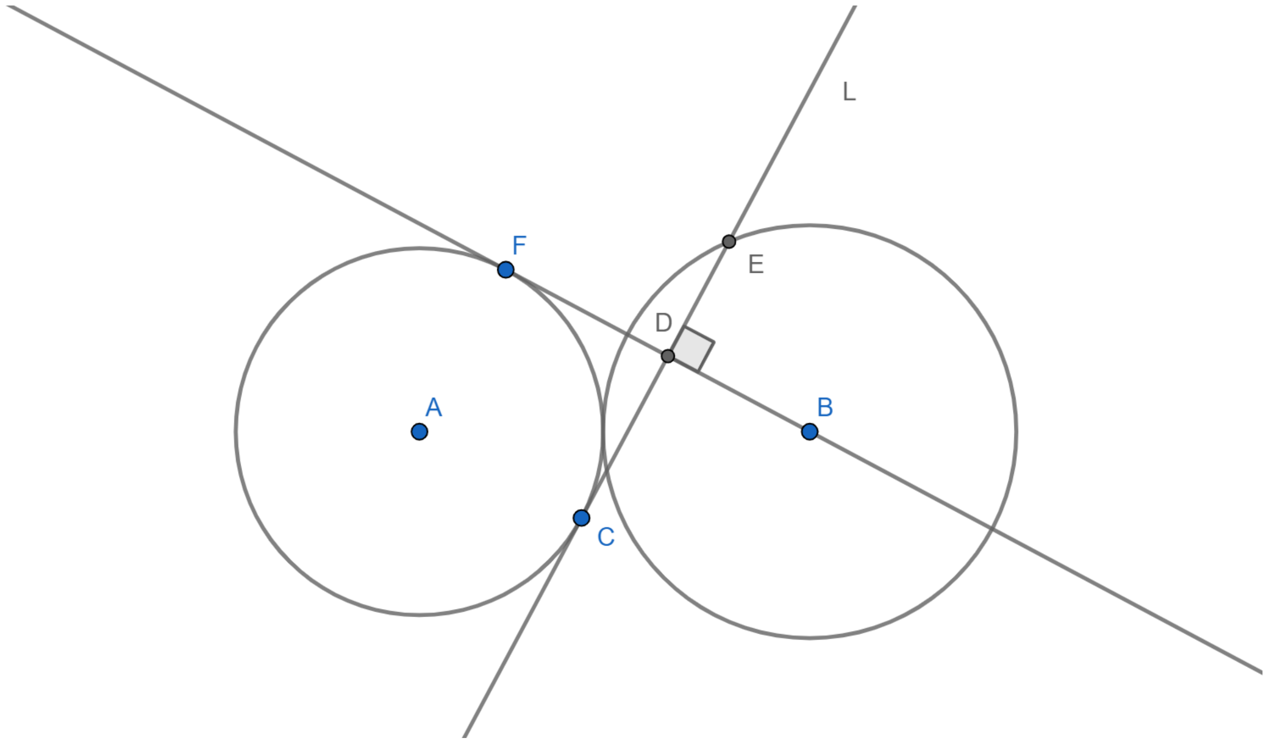
    $9^2 = 81, 99^2 = 9801, 999^2 = 998001$, from here, we can see that each one will add a 9 at the start, and add a 0 between the 8 and the 1. Therefore, for 999999^2, it will equal 999998000001. The sum of all the digits is equal to 54.

15. $5040n$ has a square number of factors, where n is a positive integer. What is the sum of the digits of the lowest possible value of n?

    Answer: $\boxed{9}$

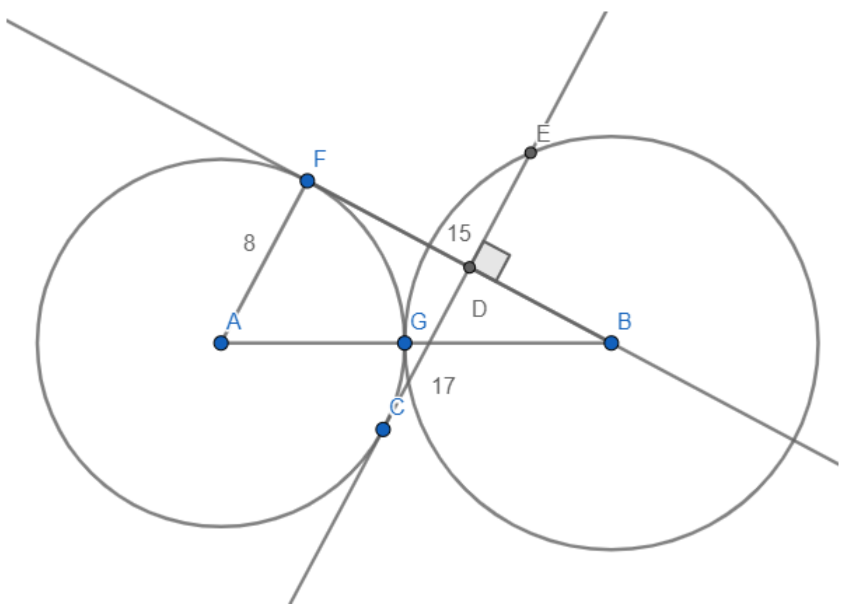
    $5040 = 2^4 \cdot 3^2 \cdot 5^1 \cdot 7^1$. We see there will be $(4+1)(2+1)(1+1)(1+1) = 5 \cdot 3 \cdot 2 \cdot 2$ factors. The 2s pair off to make a square, so you only have to increase the power of the 3s by 2 to make there a square number of factors ($5 \cdot 5 \cdot 2 \cdot 2 = 100$ factors), thus $3^2 = 9$.

16. Two circles with centres A and B with radii of length 8 and 9, respectively, are externally tangent. A line L drawn through B is tangent to circle A. A second line is drawn such that it is perpendicular to line L and tangent to circle A. The points are then labeled as shown below. What is DE^2?



Answer: $\boxed{32}$

We see that $\overline{AF} = 8$ and $\overline{AB} = 8 + 9 = 17$. Then through the Pythagorean theorem, we see $\overline{FB} = 15$, so $\overline{DB} = 15 - 8 = 7$. Then the distance from D to the far end of the circle is $7 + 9 = 16$, and the remaining bit has length $18 - 16 = 2$, so by power of a point $DE^2 = 2 \cdot 16 = 32$



17. Define two functions $f(x)$ and $g(x)$ as follows:

$f(x) = \sqrt{9x^2 + 4y^2 + 24x + 16y + 12xy + 16}$
$g(x) = 2x - 2y + 2xy - 1 - x^2 - y^2$

There is one unique set of real numbers x and y such that $f(x) = g(x)$. What is $3x + 2y$?

Answer: $\boxed{-4}$

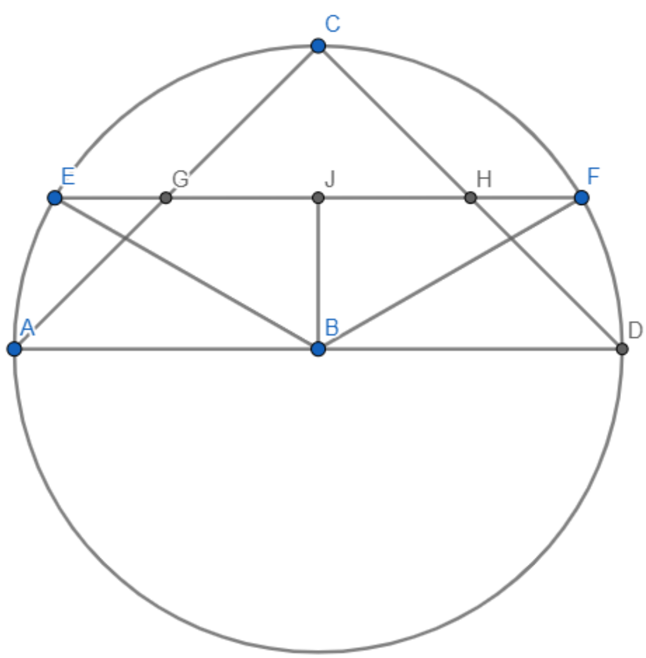
$f(x)$ can be factored into a square within a root, resulting in $\sqrt{(3x + 2y + 4)^2}$ Of course, this must be greater or equal to 0.

Next, $g(x)$ can be factored into $-(x - y - 1)^2$ This must be less than or equal to 0. As the problem states, $f(x) = g(x)$, which means that both $f(x)$ and $g(x)$ are zero. From there, it is trivial to solve the system of equations for x and y. The answer is -4, with $x = -\frac{2}{5}$ and $y = -\frac{7}{5}$.

18. $\triangle ABC$ with side lengths $AB = 2\sqrt{2}$ and $BC = 2\sqrt{2}$ is inscribed in a circle with diameter 4. A chord of length $2\sqrt{3}$, parallel to the chord AC, divides the triangle into two shapes: a trapezoid and a triangle. What is the area of this new triangle?

Answer: $\boxed{1}$

When we draw it out, we can see that $BJ = 1$ and that $\delta GHC \sim \delta ADC$. We know $BC = 2$, so the two triangles are similar by a factor of 2 and the area of δGHC is $\frac{1}{4}$ the area of the bigger triangle. The larger triangle has area $\frac{4 \cdot 2}{2} = 4$, so the area of δGHC is 1.



19. There is a polynomial $P(x) = x^3 + Ax^2 + Bx + C$, where all coefficients are rational numbers. If $P(2 + i) = 0$ and $B = 13$, what is the value of A?

Answer: $\boxed{-6}$

All coefficients are rational, and $2+i$ is a root. According to the complex conjugate theorem, $2-i$ must also be a root. Set the unknown root as n, giving $P(x) = (x-(2-i))(x-(2+i))(x-n)$. Expand this and use undetermined coefficients to find out value of $n = 2$ from $B = 13$. Then A is easily solved.

20. Ashachu starts adding and gets bored of the small sums they get from adding positive integers together, so he starts turning the $+$ signs into $\times$'s. For instance:

$$5+3+2 = 10 \longrightarrow 5 \times 3 \times 2 = 30$$

He finds that the expression they create often has a greater value than the original sum. Let N be the greatest value Ashachu can make using an expression with sum 2022. What are the last two digits of N?

Answer: $\boxed{69}$

Through trial and error we see that any nummber that is 4 or greater can be split into 2's and 3's such that their product equals or is greater than the original number (e.g. $5 = 2+3$. $2 \times 3 = 6 > 5$). Thus we should be splitting the 2022 into either 2's or 3's. We just have to compare 2^1011 and 3^674. We see that they are equivalent to 8^337 and 9^337, respectively, so we need to find 3^674. Using Chinese Remainder Theorem, we find this value to be 69 (mod 100).