2021 Team Math Attack Contest

Team Contest

December 18, 2021

1 Part 1

1. How many different four-letter sequences can be made by rearranging the letters of "MATH"?

   ANSWER: $\boxed{24}$

   Because there are only 24 permutations, this problem could also be solved by writing them all out.

2. In poker, a "Four of a Kind" is a hand of 5 total cards, consisting of 4 cards with the same rank but different suits, and a 5th card that could be anything. How many Four of a Kind hands are possible in a standard deck of 52 cards?

   ANSWER: $\boxed{624}$

   For the first four cards: 13C1 different ranks x 4C4 different suits = 13 x 1

   For the last random card: After taking 4 cards to be the four of a kind, there is only 48 other cards to choose from to be the last card.

   Answer = 13 x 1 x 48 = 624

3. What is the middle digit of 1111111^2?

   ANSWER: $\boxed{7}$

   Observe the following pattern with squares of a series of 1s:

   $11^2 = 121$

   $111^2 = 12321$

   $1111^2 = 1234321$

   Following this pattern, seven 1s squared would have a middle digit of 7

4. My birthday is in the latter half of the year and in 2019 it happened to be on a Tuesday. When will be the next year my birthday occurs on a Wednesday?

   ANSWER: $\boxed{2025}$

   2019 = Tuesday

   2020 (leap year) = Thursday

   2021 = Friday

2022 = Saturday

2023 = Sunday

2024 (leap year) = Tuesday

2025 = Wednesday

5. Richard has 4L of a 20% lemonade drink, and 4L of a 70% lemonade drink. How much of the 70% lemonade drink should Richard combine with the 20% lemonade drink to create 3L of a 60% lemonade drink?

   ANSWER: $\boxed{2.4L}$

   We create a system of equations Let the amount of the 20Then, we have x+y= 3, 0.2x+0.7y = 0.6*3 So, we need 2.4L of the 70

Part 2

6. If $a - 7 = b + 3 = c - 2 = d + 22$, which one of a, b, c, d is largest?

   ANSWER: $\boxed{a}$

   By adding 7 to all the equations, we get: $a = b + 10 = c + 5 = d + 29$ Therefore, a is the largest.

7. A quadrilateral is built with each side being tangent to a circle inscribed in it. If two opposite sides are of length 15 and 20, and a third side is length 28, what is the length of the last side?

   ANSWER: $\boxed{7}$

   When there is a triangle with a circle inscribed, as the midpoint of the circle is where the angle bisector is, the two parts of two different sides that are adjacent after being "cut" by the tangent points will be the same length. Therefore the two opposite sides of the quadrilateral will have the same distance sum as the other two sides, and therefore the answer is 7.

8. The geometric mean of 2 numbers is the square root of the product of the numbers. The arithmetic mean of 2 numbers is their sum divided by 2. The sum of 2 positive integers a, b is 20. If the geometric mean of a, b is greater than or equal to the arithmetic mean of these 2 numbers, then what is ab?

   ANSWER: $\boxed{100}$

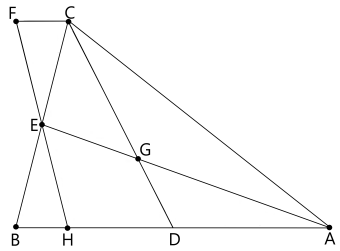
   By AM-GM $a = b = 20/2 = 10$, so $ab = 100$.

9. How many 0's are at the end of 100!? (A factorial is the product of all positive integers less than or equal to a given positive integer. For example, $7! = 1 * 2 * 3 * 4 * 5 * 6 * 7$).

   ANSWER: $\boxed{24}$

   There are 24 5's in 10! and more than 24 2's in 10!, so the number of 0's at the end of 10! will be 24.

10. In triangle $\triangle CBA$, $\overline{CD}$ and $\overline{AE}$ are medians, and $\overline{FC}$ is parallel to $\overline{AB}$. The area of $FCGE$ is 7, and the area of $EGDH$ is 11. Compute the area of triangle $\triangle CBA$.



ANSWER: $\boxed{36}$

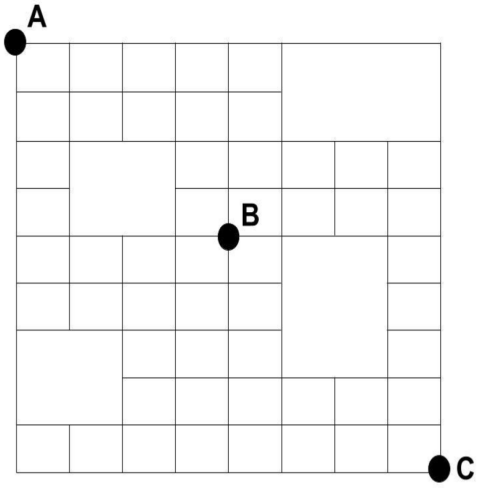
Triangle $\triangle CEF$ is congruent to triangle $\triangle BEH$ through ASA congruence.

$BCD = CEG + EGDH + BEH = FCGE + EGDH = 7 + 11 = 18$.

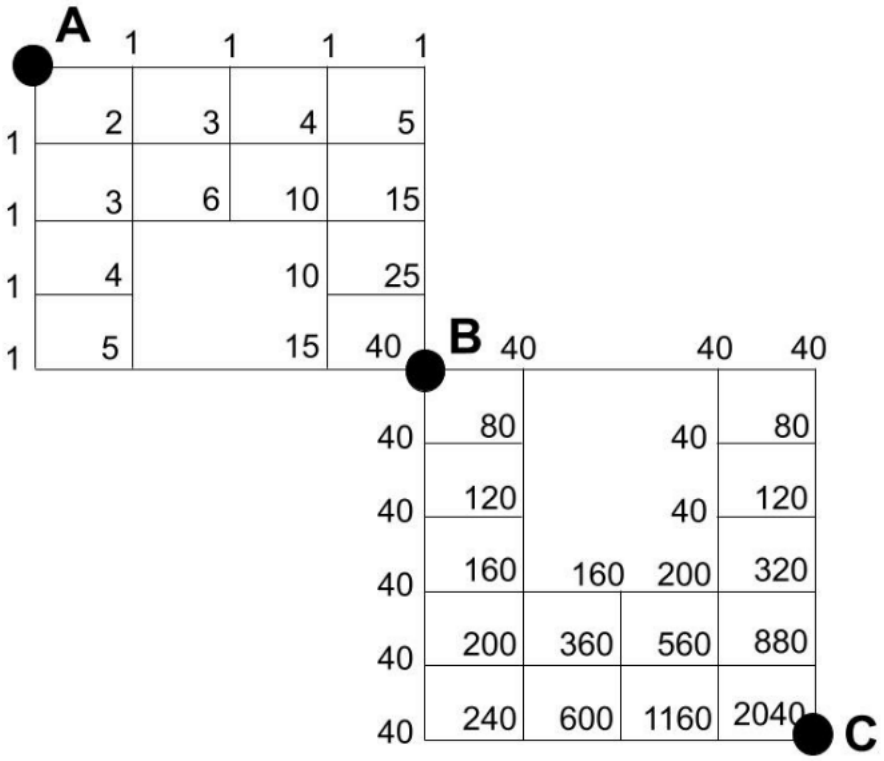
Since CD is the median, AD = BD, so the total area of CBA is simply 2 $BCD = 36$

Part 3

11. How many ways are there to travel from A to B, and then to C if you can only move down or to the right?



ANSWER: $\boxed{2040}$



12. The Fibonacci sequence is the series of numbers:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, ...$$

where the next number in the sequence is found by adding up the two numbers before it.

Every positive integer n can be expressed uniquely as a sum of distinct non-consecutive Fibonacci numbers. This is known as the Zeckendorf representation of n. For example, here is the Zeckendorf representation of 54:

$$54 = 2 + 5 + 13 + 34$$

The sum of the digits in the Zeckendorf representation of 54 is $2 + 5 + (1 + 3) + (3 + 4) = 18$. What is the sum of the digits of the Zeckendorf representation for 2021?

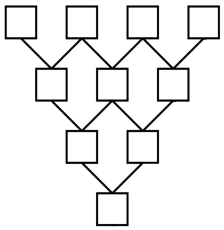
ANSWER: $\boxed{50}$

The easiest way to find the Zeckendorf representation is through the greedy algorithm, choosing the largest Fibonacci number at each stage. Listing out the Fibonacci numbers and using this method, we arrive at:

$2021 = 1597 + 377 + 34 + 13$

Therefore, the sum of the digits will be 50.

13. Four distinct integers are to be chosen from the set 1, 2, 3, 4, 5, 6, 7, 8 and placed in some order in the top row of boxes in the diagram. Each box that is not in the top row then contains the product of the integers in the two boxes connected to it in the row directly above. Determine the number of ways in which the integers can be chosen and placed in the top row so that the integer in the bottom box is 3000.



ANSWER: $\boxed{12}$

Let the numbers in the top be a, b, c, d. Then the final number can be expressed as $a*b3*c3*d$. Then, let's factorize 3000. 3000 = 53*23*3. so either b or c is 5. And either a or d is 3. Then we either have 8 in the position of a or d and 1 in the position of b or c. Or, we have 2 in the position of b or c, and 1 in the position of a or d. But there is also another solution with a and d as 6 and 4, and b and c being 1 and 5. So, the solution set (a, b, c, d) can be (1,2,5,3) or (8,1,5,3) or (6,1,5,4). But since a and d can be swapped and so can b and c, each solution has 4 permutations, for a total of 12 acceptable ways to fill the top row

14. Richard, Henry, and Michelle have 14 books in total. They each have at least one book, each one has a positive number of books, and are initially only aware of their own number of books.

Richard: I can tell using the number of books I have that the number Henry and Michelle have are not the same

Henry: I can tell using the number of books I have that there is no way any of us have the same number of books.

Michelle: After hearing what Richard and Henry said, I can use my number to figure out how many books we each have.

How many books do they each have? Answer with Michelle first, Richard second, and Henry last. For example, if Michelle has 3 books, Richard 2, and Henry 1. Submit as 321

ANSWER: $\boxed{617}$

After what H said, we know that in order to guarantee that M and R are different, M+R must be odd, and in order to know M, R, and H are different, H must be an odd number greater than or equal to 7.

So H = 7, 9, 11, 13

After what R said, we know that R must be odd, because 14 - odd = odd, which means H+M must be odd, making one even and one odd. If R was even, it is possible that H and M could be the same number. Since H is greater than or equal to 7, then R will have to be less than 7

So R = 1, 3, 5

This means that Michelle has an even number of books less than 7, and since she can determine each person's number using her number, we use possible M values to find a set of numbers in which only 1 combination is possible.

If M=2: R=1, H=11 OR R=3, H=9 OR R=5, H=7 $\rightarrow$ more than one combination

If M=4: R=1, H=9 OR R=3, H=7 $\rightarrow$ once again, not possible

If M=6: R=1, H=7 $\rightarrow$ only one combination is possible, so this is the correct combination

15. A machine will do the following 3 steps if inputted with a number x:

    1. Replace the first two digits of the number with the sum of the two first digits.
    2. Replace the last two digits of the number with the sum of the last two digits.
    3. Repeat until only a single number remains.

    Ex. If the number 98123 is inputted into the machine, it will change as such:

    $$98123 \rightarrow 17123 \rightarrow 1715 \rightarrow 815 \rightarrow 86 \rightarrow 14 \rightarrow 5$$

    What will be the result if 2021202220232024202520262027202820 29 is inputted into the machine?

    ANSWER: $\boxed{9}$

    Method 1: Brute force

    Method 2: Recall that one can sum the digits of a number to get the result of its modulo 9 congruent result. It's important to note that you don't necessarily need to sum all the digits of a number to get its modulo 9 congruent result. We can recognize that for any number inputted into the machine, it will share the same modulo 9 as the resulting number because the machine's operations consist only of summing various digits together. Note that it doesn't even matter which digits the machine chooses to sum together in regards to the final result, and that the choice to sum either the first/last 2 digits is completely arbitrary (meant only to fool unsuspecting mathletes). Thus, the final output of the machine will always be the 1-digit non-zero result when taking modulo 9 of the input number. Note that the result must be non-zero so long as the input number is non-zero because the machine's summation operations will always preserve at least one non-zero digit.

    To easily take the modulo 9 of the number given in the question, we can iterate through summing together all of its digits. After one iteration, we get 81. We may choose to sum the digits of this resulting number once more, but it is also trivial to see that $81 \pmod 9 = 9$, which is the final answer.