Round 1

Round 1 Question 1

A teacher selects 3 students from 5 boys and 4 girls to form a Team Math Attack team. The team must have at least one boy and one girl.

There are *A* distinct ways to form the team. What is *A*?

Solution:

Since there will be 3 students, it will either be 1 girl and 2 boys, or 1 boy and 2 girls.

Case 1: 1 girl and 2 boys

Here we need to choose 1 girl from 4 girls, and 2 boys from 5 boys.

So, the number of ways to form a team in this case is:

$$\binom{4}{1} \times \binom{5}{2} = \frac{4!}{1! \times (4-1)!} \times \frac{5!}{2! \times (5-2)!} = 40$$

Case 2: 1 boy and 2 girls

Here we need to choose 2 girls from 4 girls, and 1 boy from 5 boys.

So, the number of ways to form a team in this case is:

$$\binom{4}{2} \times \binom{5}{1} = \frac{4!}{2! \times (4-2)!} \times \frac{5!}{1! \times (5-1)!} = 30$$

So, the overall number of ways to form a team is $40 + 30 = 70$ ways

$A = 70$

Round 1 Question 2

A pair of shoes are being sold for *A*$ (where *A* is the answer to Question 1). The store decides to offer these shoes for 10% off. After the discount, the store also makes 50% profit on the buying price.

Let the buying price be *B*$. What is *B*?

Solution:

Recall that *A* is 70, so the shoes are initially being sold at 70$

After the 10% discount, the shoes are now sold at 63$

If the store makes 50% profit, and the buying price is *B* then we have the following relation:

$$B + 50\% \times B = 63\$$$

Simplifying this, we have:

$$1.5B = 63\$$$

$$B = \frac{63\$}{1.5} = 42\$$$

$$B = 42\$$$

Round 1 Question 3

A 10m × 10 m flat square roof receives $B \div 10$ mm of rainfall (where $B \div 10$ is the answer to Question 2 divided by 10). All of this water (and no other water) drains into an empty container in the shape of a cube with side lengths of 1.0 m.

After all the rain has entered the barrel, it is *C*% full. What is *C*?

Solution:

Recall that B is 42, so $B \div 10$ is 4.2, so the amount of rain being received is 4.2 mm.

We can imagine that the amount of rain received is a rectangular prism with length and width of 10 m, and a height of 4.2 mm.

Thus, the volume of rain received is:

$$10m \times 10m \times 4.2mm$$

Recalling that $1m = 1000mm$, we have:

$$10m \times 10m \times \frac{4.2}{1000}m = 0.42m^3$$

Since the volume of the cube is $1m^3$, the percent of the cube occupied by the water is:

$$\frac{0.42m^3}{1m^3} \times 100\% = 42\%$$

$$C = 42\%$$

Round 2

Round 2 Question 1

Marie and Bob live in separate towns, and are both driving to their parents' house for dinner. Marie has to travel 250 km and Bob has to travel 300 km. Bob is going 15 km/h faster than Marie. Marie and Bob both leave their houses at the same time, both travel at a constant speed, and both reach their destination at the same time.

Let *D* be the **units digit** of Marie's speed. What is *D*?

Solution:

Let M be Marie's speed, and B be Bob's speed.

Recall that $speed = \frac{distance}{time}$, so $time = \frac{distance}{speed}$

Since Bob travels 15 km/h faster than Marie, and they take the same amount of time, we have:

$$\frac{250}{M} = \frac{300}{B} \ and\ B = M + 15$$

Substituting the second equation into the first gives us:

$$\frac{250}{M} = \frac{300}{M+15}$$

$$250M + 3750 = 300M$$

$$50M = 3750$$

$$M = 75$$

So, *D*, which is the units digit of *M*, is 5.

$$D = 5$$

Round 2 Question 2

The mathematical mean is the sum of all numbers in a series divided by the number of terms in that series. The mode is the most common value in the series. Below is a series of numbers:

$$\{4, 2, 9, 5, 8, 6, 7, X, Y, Z\}$$

Let E be the product of X, Y, and Z, such that both the mean and mode of this series is D (where D is the answer to Question 1). What is E?

Solution:

Recall that D is 5, so the mean and mode of the series is 5.

Since 5 is the mean, we have:

$$\frac{4+2+9+5+8+6+7+X+Y+Z}{10} = 5$$

$$41 + X + Y + Z = 50$$

$$X + Y + Z = 9$$

Since the mode is also 5, one of $X,\ Y,\ and\ Z$ is 5

We suppose $X = 5$ (note that order does not matter in this case). Then we see that

$$(X, Y, Z) = (5, 2, 2)\ or\ (5, 1, 3)$$

However, if $(X, Y, Z) = (5, 2, 2)$, 2 would be the mode, not 5. So, $(X, Y, Z) = (5, 1, 3)$

$$E = X \times Y \times Z = 5 \times 1 \times 3 = 15$$

$E = 15$

Round 2 Question 3

Steven makes a grid. Column 1 has the first *E* positive even numbers written out (where *E* is the answer to Question 2). Each row is then filled with the multiples of the first number in that row, as demonstrated below:

2	4	6	8	10	…
4	8	12	16	20	…
6	12	18	24	30	…
8	16	24	32	40	…
10	20	30	40	50	…
…	…	…	…	…	…

Let *F* be the sum of all numbers in the 62nd column. What is *F*?

Solution 1:

Recall that *E* is 15, so there are 15 even numbers in the first column.

We notice that the n^{th} row has the same number as the n^{th} column in the example grid. Then, if instead of stopping at the 15th even number, we extended the graph out infinitely, this relation would always hold true.

Thus, if we wish to find the sum of the 15 numbers in the 62nd column, it is sufficient to find the sum of the first 15 numbers in the 62nd row of the extended grid. Since the rows are created by listing out the multiples of the starting number, we need only to find the sum of the first 15 multiples of the 62nd even number, which is $62 \times 2 = 124$.

The sum we desire is:

$$124 + 2 \times 124 + 3 \times 124 + ... + 15 \times 124$$

$$= 124 \times (1 + 2 + 3 + ... + 15)$$

$$= 124 \times \frac{15\times(15+1)}{2}$$

$$= 124 \times 120 = 14880$$

$$F = 14880$$

Solution 2:

Recall that *E* is 15, so there are 15 even numbers in the first column.

First, we notice that the n^{th} multiple of 2 is $n \times 2$, so the 15th multiple of 2 is 30.

We notice that in the n^{th} multiple of any even number listed in the first column occurs in the n^{th} column, and is equal to the initial number multiplied by n.

Then, we have that the sum of all numbers in the 62nd column is:

$$2 \times 62 + 4 \times 62 + ... + 30 \times 62$$

$$= 62 \times (2 + 4 + ... + 30)$$

$$= 62 \times 2 \times (1 + 2 + ... + 15)$$

$$= 124 \times (1 + 2 + ... + 15)$$

Just like above, this equals 14480

$$F = 14880$$

Round 3

Round 3 Question 1

Let $f(f(3)) = 31,\ f(f(4)) = 40$, and $f(f(5)) = 49$ for some polynomial $f(x)$.

Let G equal $-\ f(-\ 0.5)$. What is G?

Solutions:

Let $g(x)$.

We notice that $g(3) + 9 = g(4), and\ g(4) + 9 = g(5)$. This means that $g(x)$ is a linear function with a slope of 9.

We can express $g(x)$ as $9x + b$ for some number b.
$g(3) = 31,\ so\ 9 \times 3 + b = 31,\ so\ b = 4.$

Thus, $g(x) = 9x + 4$.

Since $g(x)$ is linear, $f(x)$ is linear as well. Why is this the case?

> Suppose $f(x)$ contains the term ax^n for some numbers a and n, where n is greater than 0. Then, $f(f(x))$ would have to contain the term $a\left(ax^n\right)^n = a^{n+1}x^{n^2}$. Since $f(f(x))$ is linear (recall that $f(f(x)) = g(x) = 9x + 4$) then $a^{n+1}x^{n^2}$ must be a linear term as well. So, n^2 must be 1. Since $n \geq 1,\ n = 1$, and every term in $f(x)$ is either a constant or in the form ax.

Now that we know $f(x)$ is linear, let $f(x) = cx + d$ for some numbers $c\ and\ d$.

$$f(f(x)) = c(cx + d) + d = c^2x + cd + d = 9x + 4$$

So, $c = \pm 3$

Case 1: $c = 3$

> $3d + d = 4$, so $d = 1$

Case 2: $c =- 3$

> $-\ 3d + d = 4$ so $d = -\ 2$

Thus, $f(x) = 3x + 1$ or $f(x) = -\ 3x - 2$

$$-\ f(-\ 0.5) =- (3 \times - 0.5 + 1) = -(-\ 3 \times - 0.5 - 2) = 0.5$$

$G = 0.5$

Round 3 Question 2

Ling writes down $G \times 10$ consecutive whole numbers (where $G \times 10$ is the answer to Question 1 multiplied by 10). She adds the squares of these numbers together and gets 23815 as the sum.

Let *H* be the sum of the largest and smallest of these numbers. What is *H*?

Solution:

Recall that $G = 0.5$, so $G \times 10 = 5$.

We let the middle number be x

Then, the sum is:

$$(x - 2)^2 + (x - 1)^2 + x^2 + (x + 1)^2 + (x + 2)^2$$

Expanding this we have:

$$\left(x^2 - 4x + 4\right) + \left(x^2 - 2x + 1\right) + x^2 + \left(x^2 + 2x + 1\right) + (x^2 + 4x + 4)$$

$$= 5x^2 + 10$$

Now we have that $23815 = 5x^2 + 10$. Solving for x, we have:

$$5x^2 = 23805$$

$$x^2 = 4761$$

$$x = \pm 69$$

However, since Ling has written down consecutive whole numbers, so $x \geq 2$, so $x = 69$

The smallest number and largest numbers added together are $(69 - 2) + (69 + 2) = 138$

$H = 138$

Round 3 Question 3

An arithmetic sequence is a set of numbers $\{a_1, a_2, ..., a_{n-1}, a_n\}$ such that $a_n = a_{n-1} + t$ for some number t. For example, $\{1, 3, 5, 7, 9\}$ would be an arithmetic sequence.

A certain arithmetic sequence has the property $a_{20} + a_{21} = a_{2021} = H$ (where H is the answer to Question 3).

Let $I = a_1$. What is I if a_1 is a fraction written in lowest terms?

Solution:

Recall that $H = 138$.

Since
$a_n = a_{n-1} + t,\ we\ have\ a_2 = a_1 + t,\ a_3 = a_2 + t = a_1 + 2t,\ a_4 = a_3 + t = a_1 + 3t$

We begin to see a pattern: $a_n = a_1 + (n - 1)t$

We know that $a_{20} + a_{21} = 138$. Extrapolating from this gives:

$$a_{20} + a_{21} = a_1 + 19t + a_1 + 20t = 2a_1 + 39t$$

We also know that $a_{2021} = 138$. Extrapolating from this gives:

$$a_{2021} = a_1 + 2020t$$

So, we have that $2a_1 + 39t = 138$ and $a_1 + 2020t = 138$. From this we have

$4040a_1 + 39 \times 2020t = 138 \times 2020$ and $39a_1 + 39 \times 2020t = 138 \times 39$

Subtracting the two equations gives $4001a_1 = 138 \times (2020 - 39)$

So, $a_1 = \frac{273378}{4001}$

$$I = \frac{273378}{4001}$$