0 Practice Round

0.1 Problem 1

A leap year happens on a year if that year is divisible by 4, unless the year is a multiple of a hundred. However, if it is a multiple of 400, then it is a leap year. When is the next leap year (after 2020)?

ANSWER: $\boxed{2024}$

2021, 2022, and 2023 are all not leap years. 2024 is divisible by 4 and is not a multiple of 100, so $\boxed{2024}$ is the next leap year.

0.2 Problem 2

Let x be the answer to Problem 1.

The area of a square is $x + 1$. What is the sidelength of this square?

ANSWER: $\boxed{45}$

$\sqrt{2024 + 1} = \boxed{45}$

0.3 Problem 3

Let y be the answer to Problem 2.

Michael is reading a short story with pages numbered from 1 to y. How many odd-numbered pages are there?

ANSWER: $\boxed{23}$

Since 45 is odd, we start and end on odd-numbered pages, so there must be 22 even-numbered pages and $\boxed{23}$ odd-numbered pages.

1 Round 1

1.1 Problem 1

What is $1 - 2 + 3 - 4 + 5 - 6 + \cdots + 99 - 100 + 101$?

ANSWER: $\boxed{51}$

We can group the terms as follows:

$$\begin{aligned}1-2+3-4+\cdots+99-100+101 &= (1-2)+(3-4)+\cdots+(99-100)+101\\ &= (-1)+(-1)+(-1)+\cdots+(-1)+101\\ &= (-1)\times 50+101\\ &= \boxed{51}.\end{aligned}$$

1.2 Problem 2

Let x be the answer to Problem 1.

Cheryl has \$x. She gives a third of her money to her mom and half of the remaining amount to her sister. Finally, she makes \$8 from selling lemonade. How much money does Cheryl have left?

ANSWER: $\boxed{\$25}$

Method 1:

She gives $\frac{1}{3}\times \$51 = \$17$ to her mom, so she has $\$51 - \$17 = \$34$ left. She gives $\frac{1}{2}\times \$34 = \$17$ to her sister, so she has $\$34 - \$17 = \$17$ left. Finally, she makes \$8, so she ends up with $\$17 + \$8 = \boxed{\$25}$ left.

Method 2:

After giving a third of her money away, Cheryl has $\frac{2}{3}x$ remaining. She gives away half this amount, so then she has $\frac{1}{2}\times\frac{2}{3}x = \frac{1}{3}x$ of her money remaining. Finally, she makes \$8, so she ends up with $\frac{1}{3}x + \$8$. Plugging in $x = \$51$ gives us $\frac{\$51}{3} + \$8 = \$17 + \$8 = \boxed{\$25}$.

1.3 Problem 3

Let y be the answer to Problem 2.

Larry and Harry each have a toy motor. Larry builds a toy car with his motor that travels 10 meters in 5 seconds. Harry also builds a toy car, but his travels 10 meters in $\frac{y}{3}$ seconds. Assuming that adding the motors is the same as adding their speeds, how long (in seconds) will it take to travel 64 meters on a car that has both Harry and Larry's motors on it?

ANSWER: $\boxed{20}$

Method 1:

Since speed is equal to distance divided by time, Larry's motor has a speed of $10\div 5 = 2$ meters per second and Harry's motor has a speed of $10\div(\frac{25}{3}) = \frac{6}{5}$ meters per second, so in total their combined motors travels at $\frac{16}{5}$ meters per second. Rearranging the equation "speed equals distance over time", we get "time equals distance over speed", so it takes $64\div\frac{16}{5} = \boxed{20}$ seconds total to travel 64 meters.

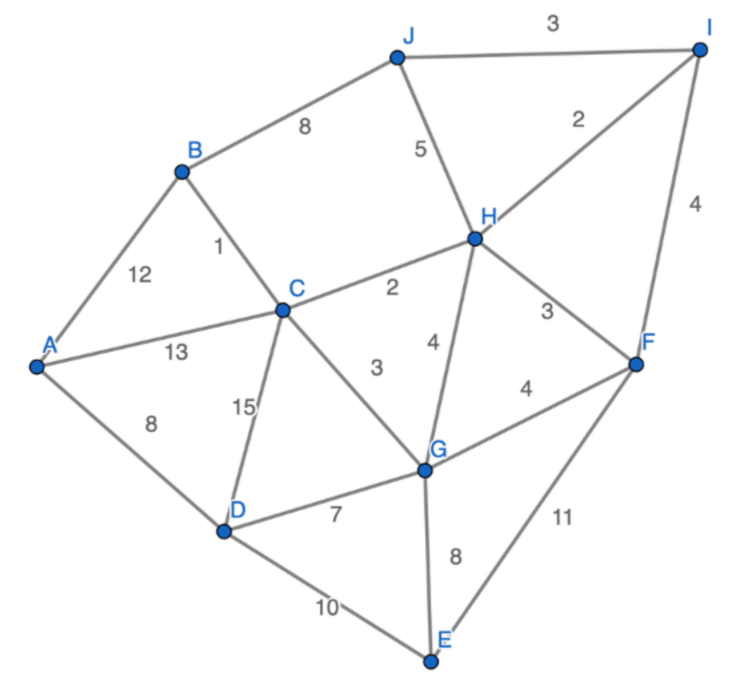
Method 2:

Larry's motor has a speed of 10 meters $\div$ 5 seconds $= 2\frac{m}{s}$. Harry's motor has a speed of 10 meters $\div$ $\frac{y}{3}$ seconds $= \frac{30}{y}\frac{m}{s}$. Adding these together gives us a combined speed of $(\frac{30}{y} + 2)\frac{m}{s} = \frac{30+2y}{y}\frac{m}{s}$, so to travel 64 meters will take $64 \div \frac{30+2y}{y} = \frac{64y}{30+2y}$ seconds. Plugging in $y = 25$ gives us $\frac{64\times 25}{30+2\times 25}s = \boxed{20}$ seconds.

2 Round 2

2.1 Problem 1

Raiyana is driving through a city with many bridges, shown as lines below. To cross a bridge, she must pay its toll, indicated by the numbers on the graph. What is the smallest amount she can pay if she starts at point A, visits each city exactly once, and returns to point A?



ANSWER: $\boxed{57}$

By starting at A and following the cities in the order A, B, C, H, J, I, F, G, E, D, A, we can achieve a cost of

$12 + 1 + 2 + 5 + 3 + 4 + 4 + 8 + 10 + 8 = \boxed{57}$.

2.2 Problem 2

Let x be the answer to Problem 1.

Berry owns a furniture store with several employees. One day, Berry realizes that the average number of items sold per employee up to that point was x. The very next day, Jerry sold 6 items, Andrew sold 7, Ryan sold 8, and the rest all sold 2 items each. This raised the average number of items sold per employee to 60. How many employees work at Berry's furniture store?

ANSWER: $\boxed{15}$

Method 1:

Let Berry have n employees.

In total, up to that day, Berry's employees have sold $57n$ items. The next day, they sell $6+7+8+(n-3)\times 2$ more, for a total of $15+59n$ products. Since the average is now 60, we have $\frac{15+59n}{n} = 60$, so $15+59n = 60n$ and $n = \boxed{15}$.

Method 2:

Let Berry have n employees.

In total, up to that day, Berry's employees have sold xn items. The next day, they sell $6+7+8+(n-3)\times 2$ more, for a total of $21+2n-6+xn = 15+(x+2)n$ items. Since the average is now 60, we have:

$$\begin{aligned} \frac{15+(x+2)n}{n} &= 60 \\ 15+(x+2)n &= 60n \\ 15 &= (60-(x+2))n \\ n &= \frac{15}{58-x}. \end{aligned}$$

Plugging in $x = 57$ gives us $\frac{15}{58-57} = \boxed{15}$.

2.3 Problem 3

Let y be the answer to Problem 2.

One day, the boys and girls in Instructor Jerry's class decided to have a contest. To create stakes, each student brought 6 candies the next day, under the rules that whichever side wins splits up all the candy among themselves. The girls found out that if they won, they would all receive y candies. How many candies would each boy receive if the boys won?

ANSWER: $\boxed{10}$.

Method 1:

Let there be g girls and b boys. The total number of candies is $6(b+g)$. If the girls each receive 15 candies, this means $\frac{6(b+g)}{g} = 15$, so $6b + 6g = 15g$, so $g = \frac{2}{3}b$. Then, $\frac{6(b+g)}{b} = \frac{6b+6(\frac{2}{3}b)}{b} = 6 + 4 = \boxed{10}$.

Method 2:

Let there be g girls and b boys. The total number of candies is $6(b+g)$. If the girls each receive y candies, this means

$$\begin{aligned}\frac{6(b+g)}{g} &= y \\ 6b + 6g &= yg \\ (y-6)g &= 6b \\ g &= \frac{6b}{y-6}.\end{aligned}$$

Thus, the number of candies each boy would receive is:

$$\begin{aligned}\frac{6(b+g)}{b} &= \frac{6(b+\frac{6b}{y-6})}{b} \\ &= \frac{6b(1+\frac{6}{y-6})}{b} \\ &= 6 + \frac{36}{y-6}.\end{aligned}$$

Plugging in $y = 15$ gives us $6 + \frac{36}{15-6} = 6 + 4 = \boxed{10}$.

3 Round 3

3.1 Problem 1

A triangular number is defined as a positive integer that can be expressed in the form $\frac{n(n+1)}{2}$, where n is a positive integer. Let S and L be the smallest and largest triangular numbers respectively such that they can each be expressed as a sum of exactly two triangular numbers, and $S < L < 50$. What is $S + L$?

ANSWER: $\boxed{42}$

The first 10 triangular numbers are the following: 1, 3, 6, 10, 15, 21, 28, 36, 45, 55. Hence, there are only 9 triangular numbers less than 50.

Note that 45 cannot be expressed as a sum of two triangular numbers since none of the differences below are triangular numbers.

$$45 - 36 = 9$$
$$45 - 28 = 17$$
$$45 - 21 = 24$$
$$45 - 15 = 30$$
$$45 - 10 = 35$$
$$45 - 6 = 39$$
$$45 - 3 = 42$$
$$45 - 1 = 44$$

Also, 1 and 3 cannot be expressed as the sum of two triangular numbers since 1 is the smallest triangular number, and $3 - 1 = 2$, which is not a triangular number.

However, 6 and 36 can be expressed as the sum of two triangular numbers since $6 = 3 + 3$, and $36 = 21 + 15$. Therefore, $S = 6$, $L = 36$, and $S + L = \boxed{42}$.

3.2 Problem 2

Let x be the answer to Problem 1.

Howard has at least one friend with each of the following hair colours: black, brown, blue, and blonde. The number of possible combinations of hair colours is $2x$.

For example, if Howard has 8 friends, a possible combination could be 2 black, 1 brown, 3 blue, and 2 blonde, and another possible combination could be 5 black, 1 brown, 1 blue, and 1 blonde.

How many friends does Howard have?

ANSWER: $\boxed{10}$

Let n be the number of friends Howard has.

First, we "assign" a hair colour to 4 of the friends. Then, we have $n - 4$ remaining friends, and we can give them hair colours however we want, without the restriction that there must be at least one friend with each hair colour.

Since we want to separate them into 4 groups, representing the hair colours, this is the same as putting 3 dividers between them. So in total, we have $n - 4$ friends and 3 dividers, for a total of $n - 1$ "spaces". We want to choose 3 of these spaces to put the dividers in. There are $n - 1$ places for the first divider, $n - 2$ remaining places for the second, and $n - 3$ remaining places for the third. However, the order of the dividers doesn't matter, so we're counting each set of positions $3! = 6$ times. Therefore, we divide $(n-1)(n-2)(n-3)$ by 6 and set this equal to $2x$, which is 84.

Thus, we are looking for three consecutive numbers which multiply to $2x \times 6 = 12x$. Plugging in $x = 42$

shows us that the product must be 504. We see that $9 \times 8 \times 7 = 504$. Thus, since $n - 1 = 9$, $n = \boxed{10}$.

3.3 Problem 3

Let y be the answer to Problem 2.

Amanda, Bernando, and Choliver are being pursued by a horde of engineers, who are chanting "pi = 3". In order to contain the spread of these engineers, they need to cover a distance of $18y$ km along a straight road to find a Totem of UnPi'ing. They have among them a motorcycle that can only carry 2 people at a time. The motorcycle can travel at a speed of 120 km/h while each of the 3 people can walk at 8 km/h. Assuming that the motorcycle does not run out of gas and that the time taken for the motorcycle to turn or for people to get on and off is negligible, what is the shortest time (in hours) it will take for ALL 3 of them to cover $18y$ km?

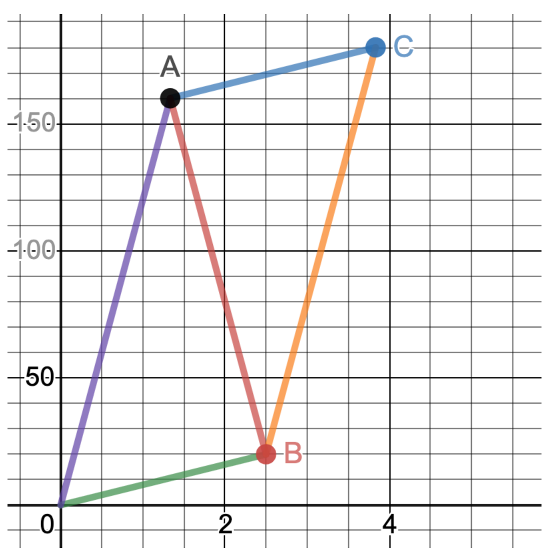
ANSWER: $\boxed{\frac{23}{6}}$.

We can solve this problem by noticing that we can achieve an optimal time if all three arrive at the end at the same time, so that no time is wasted by a teammate waiting at the end.

Let us think about the problem graphically, where time is the x-axis and distance is the y-axis. Thus, the slope of a line represents speed. Thus, we can only have slopes of ± 8 and ± 120, corresponding to the walking speed and biking speed respectively.

At the beginning, we should have Amanda and Bernando take the motorcycle towards the finish line and Choliver begin walking. Then, before the bikers reach the totem, Bernando should drop off Amanda so that she walks the remaining way. He should then bike backward to pick up Choliver and then bike toward the totem, so that all three can meet at the same time.

Thus, graphing their paths should give us a parallelogram:



See the following link for a visualization.

`https://www.desmos.com/calculator/aswbmueceu`

Let the time when Bernando drops off Amanda be a, the time when he picks up Choliver be b, and their final arrival time be c. On the graph, this means that point A is at $(a, 120a)$, B is at $(b, 8b)$, and C is at $(c, 18y)$. This gives us the following system of equations using the slopes of lines AB, BC, and AC:

$$\frac{8b - 120a}{b - a} = -120 \tag{1}$$
$$\frac{18y - 8b}{c - b} = 120 \tag{2}$$
$$\frac{18y - 120a}{c - a} = 8. \tag{3}$$

Since y is a known value (10), we have three equations and three unknowns, which we can solve:

From (1):

$$\begin{aligned}\frac{8b - 120a}{b - a} &= -120\\ b - 15a &= -15b + 15a\\ 16b &= 30a\\ b &= \frac{15}{8}a\end{aligned}$$

Combining this with (2):

$$\begin{aligned}\frac{18y - 8b}{c - b} &= 120\\ 180 - 8b &= 120c - 120b\\ 180 - 112(\frac{15}{8}a) &= 120c\\ 120c &= 180 + 210a\\ c &= \frac{6 + 7a}{4}\end{aligned}$$

Combining this with (3):

$$\begin{aligned}\frac{18y - 120a}{c - a} &= 8\\ 180 - 120a &= 8c - 8a\\ 180 - 112a &= 8(\frac{6 + 7a}{4})\\ 180 - 112a &= 12 + 14a\\ a &= \frac{4}{3}\end{aligned}$$

Finally, we substitute this back into the expression for C:

$$
\begin{aligned}
c &= \frac{6 + 7a}{4} \\
&= \frac{6 + 7\frac{4}{3}}{4} \\
&= \frac{18 + 28}{12} \\
&= \frac{23}{6}
\end{aligned}
$$

Thus, it takes the three of them $\boxed{\frac{23}{6}}$ hours to travel 180 km.