Easy

Problem 1

Jerry (a male) has 2 more sisters than he has brothers. If the total number of children in the family is divisible by 9, what is the remainder when the number of brothers in the family is divided by 3?

Solution: Let B and G be the number of boys and girls in the family, respectively. Since Jerry is a male and is not one of his own brothers $S = (B - 1) + 2$, we have

$$9k = B + G = B + (B - 1 + 2) = 2B + 1 \Rightarrow B = \frac{9k - 1}{2}$$

However, we know B must be an integer, hence $2|9k - 1$ implies $9k - 1$ is odd, and therefore k must be odd as well. Let $k = 2s + 1$ where s is a positive integer.

$$\frac{1}{2}(9k - 1) = \frac{1}{2}(9(2s + 1) - 1) = \frac{1}{2}(18s + 8) = 9s + 4$$

Upon division by 3, we find the remainder is $\boxed{1}$.

Problem 2

How many ways can you rearrange the letters in the word "CLOSENESS"?

Solution: There are 9 letters in the word "CLOSENESS", and therefore 9! ways to rearrange them all. However, we overcounted all the ways the indistinguishable E's and S's (i.e. CLOSE$_1$NE$_2$SS vs. CLOSE$_2$NE$_1$SS are counted twice). Thus we must divide by the number of ways we can arrange these indistinguishable letters. Hence the final answer is,

$$\frac{362880}{2!3!} = \boxed{30240}$$

as E appears 2 times and S appears 3 times.

Problem 3

Determine the number of times the digit 7 appears in the set $\{1, 2, \ldots, 99, 100\}$.

Solution: Case 1: 7 is in the tens digit.
There are 10 integers, namely $70, 71, \ldots, 79$.

Case 2: 7 is in the ones digit.
There are 10 integers again, namely $7, 17, 27, \ldots, 97$ for a total of $10 + 10 = \boxed{20}$. Note that we counted the 7's in 77 separately each case.

Problem 4

Find the remainder of 6^{2018} upon division by 10.

Solution: The ones digit of any positive power of 6 will be $\boxed{6}$. This can be proved by induction. Base case: $6^1 = 6$.
Inductive step; let $6^n = 10 \cdot \ell + 6$ for a positive integer n and some integer ℓ, we will prove 6^{n+1} also has a ones digit of 6. $6^{n+1} = 6 \cdot (\ell \cdot 10 + 6) = 36 + 60 \cdot \ell = 10 \cdot \ell + 6$ as desired.

Problem 5

In the set $\{49, 63, 70, 77, 126\}$, which number is the average of the other four numbers?

Solution: We claim the answer is $\boxed{77}$.

$$\frac{49 + 63 + 70 + 126}{4} = 77$$

as desired.

Medium

Problem 6

Find the units digit of $(3^{100})(2^{101})$.

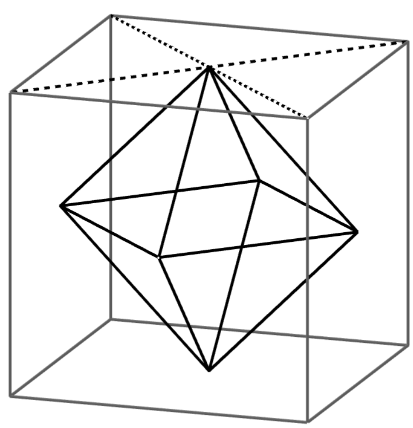
Solution: We look at each terms separately modulus 10. The residues of $3^0, 3^1, 3^2, 3^3, 3^4, 3^5, \ldots$ has corresponding residues $1, 3, 9, 7, 1, 3, \ldots$ modulus 10. We note this repeats every 4 powers. Since $100 \equiv 0 \pmod 4$, we know $3^{100} \equiv 1 \pmod{10}$.

Similarly, $2^0, 2^1, 2^2, 2^3, 2^4, 2^5, 2^6 \ldots$ has corresponding residues $1, 2, 4, 8, 6, 2, 4, \ldots$ modulus 10. Starting from 2^1 the residues repeat every 4 powers. $101 \equiv 1 \pmod 4$, hence $2^{101} \equiv 2 \pmod{10}$, (as we begin counting from 2^1).

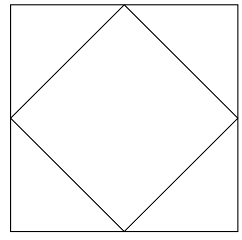
$$(3^{100})(2^{101}) \equiv (1)(2) \equiv \boxed{2} \pmod{10}$$

Problem 7

The centre of a square is the point of intersection between the two diagonals. Given a cube of side length 1, the centres of the 6 faces form the vertices of an octahedron. Find the volume of this regular octahedron.



Solution: We decompose the octahedron into two square pyramids, sharing the same square base with vertices on opposite faces. Looking at the cross section at the square base, we note the 45-45-90 triangles, where we find the side length of the inner square base to be $\frac{\sqrt{2}}{2}$.



The height of this pyramid is simply half the height of the cube. The volume of a general pyramid is $\frac{1}{3}A_{\text{base}} \cdot h$, hence the total volume of the octahedron is,

$$2 \cdot \frac{1}{3}(\frac{\sqrt{2}}{2})^2(\frac{1}{2}) = \boxed{\frac{1}{6}}$$

Problem 8

Given that $\frac{15!}{10^k} = q$ where k, q are both positive integers, determine the maximum value of k.

Solution: Recall $n! = n(n-1)(n-2)\ldots(2)(1)$. The power of 5 in the prime factorization of 15! will be smaller than the power of 2. There are only three factors in 15! divisible by 5 - namely 5, 10, 15. Hence the highest value of k is $\boxed{k = 3}$.

Problem 9

Let x be some integer. What is the sum of all solutions to the following equation,

$$((x-3)(x-4)+1)^{(x+10)(x-6)} = 1$$

Solution: Either the exponent must be 0, or the base must be 1.

Case 1: Exponent is equal to 0, or $(x+10)(x-6) = 0$
$x = -10$ or 6

Case 2: Base is equal to 1, or $(x-3)(x-4)+1 = 1$
$x = 3$ or 4.

$$-10 + 6 + 3 + 4 = \boxed{3}$$

Problem 10

In the given 7x7 grid, there is a diamond hidden in exactly 10 of the 49 squares. The numbers indicate how many adjacent squares (sharing a side or corner) contain a diamond. No square with a number also contains a diamond. Indicate the locations of these 10 diamonds in the diagram.

Solution:

			D			
	D	5	D	4	1	
	2	D	D		D	
			D	4	2	
	0			3	D	1
			D	2	1	
		D	2			

Or

			D			
	D	5	D	4	1	
	2	D	D	D		
			D	4	2	
	0			3	D	1
			D	2	1	
		D	2			

Hard

Problem 11

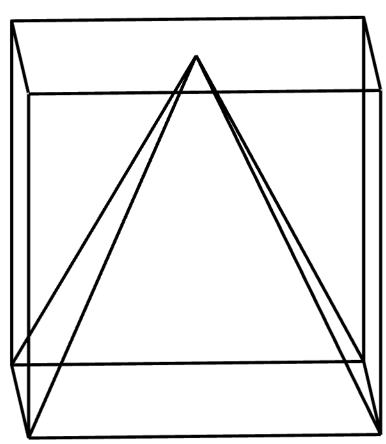
Michael and Chan are running down a 400 m circular race track at constant but different speeds. They start at the same point and run in opposite directions. Michael runs at 1.5 m/s and they pass by each other every 1 minute 40 seconds. How fast does Chan run in m/s?

Solution: Let Chan run at a rate of C m/s, since they pass by each other every 1 minute 40 s, during that time they ran a total of 400 m, we have,

$$\begin{aligned}(1.5 + C)\text{ m/s} \cdot (100\text{ s}) &= 400\text{ m}\\ \Rightarrow C &= (4 - 1.5)\text{ m/s}\\ &= \boxed{2.5\text{ m/s}}\end{aligned}$$

Problem 12

Bob has a square pyramid inside a cube that shares the same base as the cube and with the upper vertex touching the centre of the top face of the cube. Bob then decides to scale everything by a factor of 4. If the surface area of the pyramid increased by $\frac{303}{4}\text{cm}^2$, what is the side length of the cube after scaling?



Solution: Let s be the side length of the cube. We note the surface area of the pyramid is comprised of the square base and 4 congruent triangles. These triangles have a height of $\sqrt{s^2 + (\frac{1}{2}s)^2} = \frac{s\sqrt{5}}{2}$. Thus the original surface area of the pyramid is $s^2 + s\frac{\sqrt{5}}{2}$. Let S be the side length upon scaling up by 4, we know $S = 4s$.

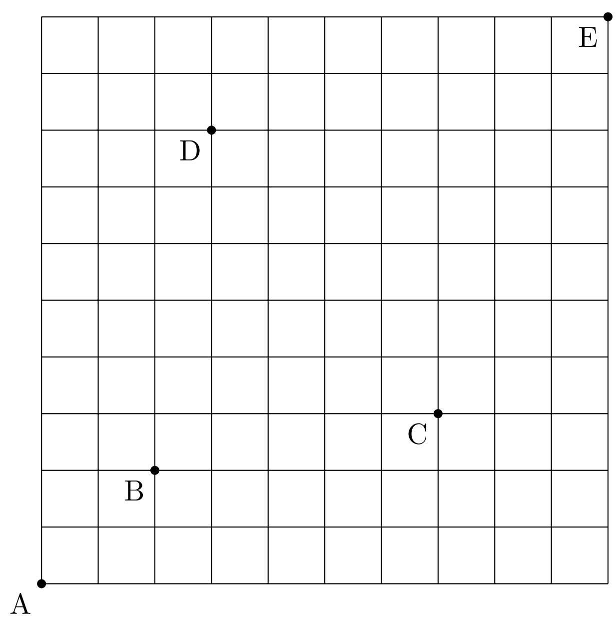
$$\begin{aligned}s^2 + s\frac{\sqrt{5}}{2} + \frac{303}{4} &= S^2 + S\frac{\sqrt{5}}{2}\\ &= 16s^2 + 4s\frac{\sqrt{5}}{2}\\ 0 &= 15s^2 + 3s\frac{\sqrt{5}}{2} - \frac{303}{4}\end{aligned}$$

We can use the quadratic equation to generate the solutions s to the equation

$$\begin{aligned} s &= \frac{-\frac{\sqrt{5}}{2} \pm \sqrt{(3\frac{\sqrt{5}}{2})^2 - 4(15)(-\frac{303}{4})}}{2(15)} \\ &= \frac{-\frac{\sqrt{5}}{2} \pm \sqrt{\frac{45}{4} + 4545}}{30} \\ &= \boxed{\frac{135 - \sqrt{5}}{60}} \end{aligned}$$

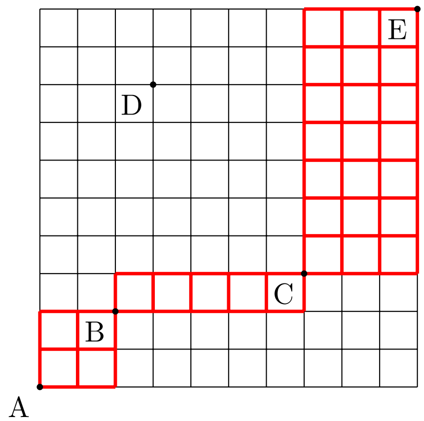
Problem 13

Jerry Li wants to move through 4 of these 5 marked points, but he can only move upwards or to the right. How many ways can Jerry fulfill these conditions?



Solution: This problem involves moving arranging a string of commands "right" and "up". It will be separated into the 4 points that Jerry passes through. If Jerry passes through C he will be unable to pass through D and vice versa. To pass through 4 points, he cannot start at any point other than A. We have two cases;

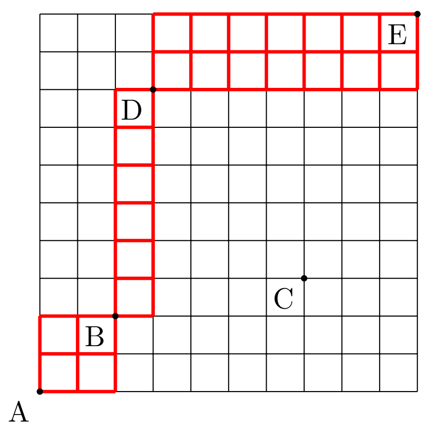
Case 1: Jerry's path is $A \to B \to C \to E$



We have 3 individual strings of "right" and "up", each of which are independent, thus the number of ways to travel this path is:

$$\binom{4}{2} \cdot \binom{6}{1} \cdot \binom{10}{3} = 6 \cdot 6 \cdot 120$$

Case 2: Jerry's path is $A \to B \to D \to E$



Using the same process, we have,

$$\binom{4}{2} \cdot \binom{7}{1} \cdot \binom{9}{2} = 6 \cdot 7 \cdot 36$$

Adding the two cases together gives $6 \cdot 6 \cdot 120 + 6 \cdot 7 \cdot 36 = \boxed{5832}$

Problem 14

The expression $343^9 \cdot 216^{12} \cdot 49^7 \cdot 2401^5 \cdot 7^6 \cdot 36^{10}$ can be written in the form $a^b \cdot c^d$, where a, b, c, d are positive integers,[1] $b < d$, and a, c are in the lowest bases possible. Find the units digit of $a^{2012} + b^{2014} + c^{2016} + d^{2018}$.

[1] On the original contest, there was a typo in the expression statement. As a result this problem was omitted.

Solution:

$$\begin{aligned}343^9 \cdot 216^{12} \cdot 49^7 \cdot 2401^5 \cdot 7^6 \cdot 36^{10} &= (7^3)^9 \cdot (6^3)^{12} \cdot (7^2)^7 \cdot (7^4)^5 \cdot 7^6 \cdot (6^2)^{10} \\ &= 7^{27} \cdot 7^{14} \cdot 7^{20} \cdot 7^6 \cdot 6^{36} \cdot 6^{20} \\ &= 6^{56} \cdot 7^{67}\end{aligned}$$

$$a = 6, b = 56, c = 7, d = 67$$

$a^{2012} = 6^{2012}$ which ends in a 6 because when you multiply a group of numbers that all end in 6, the result must end in a 6.
$b^{2014} = 56^{2014}$ which ends in a 6 because of the same reason as above.
$c^{2016} = 7^{2016}$ which ends in a 1 because if n is divisible by 4, than 7^n ends in a 1, and since 2016 is divisible by 4, 7^{2016} ends in a 1.
$d^{2018} = 67^{2018}$ which ends in a 4 because if a number A that has a units digit of 7 and N is a number where n - 2 is divisible by 4, then 7^n must end in a 9. 67^{2018} meets both of these conditions.
When determining the units digit of $a^{2012} + b^{2014} + c^{2016} + d^{2018}$, we are only concerned about adding the units digit individually for each term, and the units digit for each term is 6, 6, 1, and 9, respectively, giving a total of 22, which has a units digit of 2. Therefore, the answer is $\boxed{2}$.

Final Proof

Problem 15

The terms of the sequence $\{a_i\}$ are defined as follows

i. $a_{2n} = a_n$.

ii. For all odd n, we have $a_n = a_{n-1} + a_{n-3} + \ldots + a_2 + a_0$

Given $a_0 = 1$, prove that, for any $k \geq 2$, a_{3k} is always even and a_{3k+1}, a_{3k+2} are always odd.

Solution: We observe that for any $n = 2i + 1$ where i is non negative, we have,

$$\begin{aligned} a_n &= a_{2i+1} \\ &= a_{2i} + (a_{2i-2} + \ldots + a_2 + a_0) \\ &= a_{2i} + a_{2i-1} \\ &= a_{n-1} + a_{n-2} \end{aligned} \tag{1}$$

We prove using strong induction, we compute the first few terms for the base case.

a_{3k}	a_{3k+1}	a_{3k+2}
$a_0 = 1$	$a_1 = 1$	$a_2 = 1$
$a_3 = 2$	$a_4 = 1$	$a_5 = 3$
$a_6 = 2$	$a_7 = 5$	$a_8 = 1$
$a_9 = 6$	$a_{10} = 3$	$a_{11} = 9$

Which completes the base case.

Inductive Step: Assuming a_{3i} is even and a_{3i+1}, a_{3i+2} is odd for all $k \geq i \geq 2$. We will prove a_{3k+3} is even and a_{3k+4}, a_{3k+5} are odd.

Case 1: $3k + 3$ is odd.
Using (1) we note $a_{3k+3} = a_{3k+2} + a_{3k+1}$, both a_{3k+1}, and a_{3k+2} are odd, so a_{3k+3} is even.
$3k + 4$ is even, let $k = 2i$. Note $3k + 4 = 2(3i + 2)$, thus $a_{3k+4} = a_{2(3i+2)} = a_{3i+2}$, which is odd by our inductive hypothesis, as $3k = 6i \geq 3i + 2$.
We know $3k + 5$ is odd. Using (1) again gives $a_{3k+5} = a_{3k+4} + a_{3k+3}$. Hence a_{3k+5} is odd as desired.

Case 2: $3k + 3$ is even.
Using the same process as above, let $k + 1 = 2i$. Note $3k + 3 = 6i$, thus $a_{3k+3} = a_{6i} = a_{3i}$, which is even by our inductive hypothesis.
Using (1), we have $a_{3k+4} = a_{3k+3} + a_{3k+2}$. a_{3k+2} is odd, therfore a_{3k+4} is odd as well.
Let $k + 1 = 2i$ as before, $3k + 5 = 6i + 2$. Thus $a_{3k+5} = a_{6i+2} = a_{3i+1}$, which is odd by our inductive hypothesis. Thus we have covered all cases and our induction is complete. □

Marking Criteria:

1 - Accurately consider small cases (at least until a_8).

3 - Set up an appropriate strong induction, ensuring that the inductive step can be inferred from the base case.

1 - Considers one case of $3k + 3$ (either odd or even).

2 - Considers both cases for $3k + 3k$.

1 - Final mark awarded for complete proof with only trivial errors.

Note: An important observation is that $a_n = a_{n-1} + a_{n-2}$ for odd n. If points have not been awarded for setting up an appropriate induction, award 1 point for proving this observation.