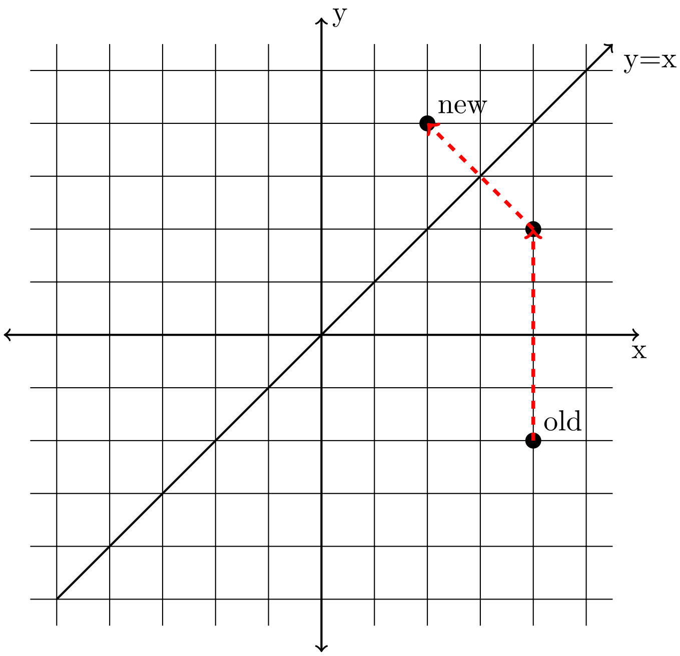
Problem 1

The point (4, -2) is reflected over the x-axis. The resulting point is then reflected again over the line with equation $y = x$. What are the coordinates of the new point?

Solution: Consider the diagram



Reflecting across x-axis changes the sign of the y component and across $y = x$ flips the x and y components around. We have:

$$(4, -2) \to (4, 2) \to \boxed{(2, 4)}$$

Problem 2

Find the sum of the first 10 positive powers of 2.

Solution: This is the sum of a geometric series, where we used the closed form: $a + ar + ar^2 + \ldots + ar^n = a(\frac{r^{n+1} - 1}{r - 1})$.

$$2^1 + 2^2 + \ldots + 2^{10} = 2 + 4 + \ldots + 1024 = 2\frac{2^{10} - 1}{2 - 1} = \boxed{2046}$$

Problem 3

The mean of the 9 data values

$$20, 80, x, 60, 15, 35, 95, 100, 70$$

is equal to $x - 5$. Find the value of x.

Solution:

$$\begin{aligned} x - 5 &= (20 + 80 + 60 + 15 + 35 + 95 + 100 + 70 + x)/9 \\ 9x - 45 &= 475 + x \\ 8x &= 520 \\ x &= \boxed{65} \end{aligned}$$

Problem 4

Monika has three distinct dogs in a line. A golden retriever, a pug, and a dalmatian. She has four different collars. How many ways can she arrange her dogs with exactly one collar each?[1]

Solution: We can potentially place 4 collars on the first dog, 3 on the second, and 2 on the third. Finally note that the dogs are distinct, and we can rearrange the three types of dogs in 3! ways. $4 \cdot 3 \cdot 2 \cdot 3! = \boxed{144}$.

Problem 5

Let a, b be the remainders of 12345678987654321 upon division by 9 and 11 respectively. Find $a + b$.

Solution: The remainder of a upon division by 9 is equal to the remainder upon dividing the sum of the digits of a. The remainder upon division by 11 is the alternating sum of the digits of a. Let $rem(a, b)$ be the remainder of a upon division by b. For division by 9,

$$\begin{aligned} rem(12345678987654321, 9) &= rem(1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 \\ &\qquad + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1, 9) \\ &= rem(81, 9) = 0 \end{aligned}$$

For division by 11,

$$\begin{aligned} rem(12345678987654321, 11) &= (1 - 2 + 3 - 4 + 5 - 6 + 7 - 8 + 9 - 8 + 7 - 6 + 5 - 4 + 3 - 2 + 1, 11) \\ &= rem(1, 11) = 1 \end{aligned}$$

Thus the sum of the two remainders is $0 + 1 = \boxed{1}$

[1] Clarified from original problem

Medium

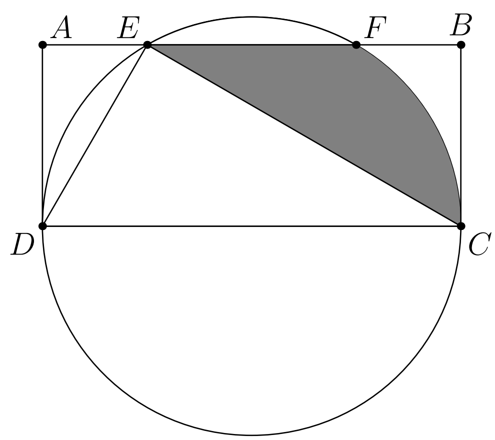
Problem 6

Jack can create an entire math contest in 10 days. Andrew can create the same math contest in 8 days. Assuming they always work at the same rate, how many days will it take for them to create one math contest if they work together?

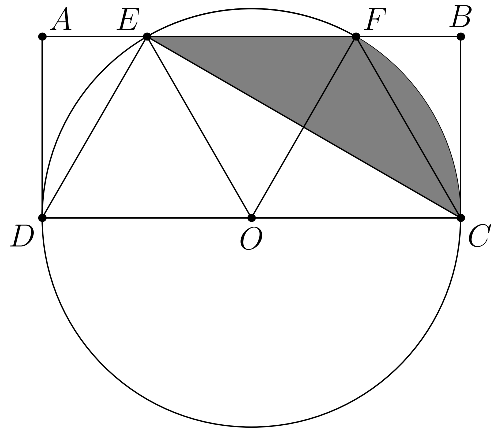
Solution: Jack and Andrew can complete a math contest at a rate of $\frac{1}{10}$ and $\frac{1}{8}$ math contests per day, respectively. Thus the combined rate they write math contest per day is $\frac{1}{10} + \frac{1}{8} = \frac{9}{40}$. Therefore they can complete one math contest in $\boxed{\frac{40}{9} \text{ days}}$.

Problem 7

In rectangle $ABCD$, point E is chosen on side AB such that $\triangle DEC$ is a right triangle. The circumcircle of $\triangle DEC$ intersects AB at F. Given $\angle DCE$ is $30°$, find the ratio of the area of the region contained inside EC, FE and the arc FC to the area of $\triangle DEC$.



Solution: Since $\angle DEC = 90°$, we know the DC is a diameter of the circumcircle. Let the centre of the circumcircle be O. $\triangle DEC$ is a 30-60-90 triangle, hence $\angle EOD = \angle CDE = 60°$. On the other side we have $\angle EOC = 180° - (30° + 30°) = 120°$. Note that E and F are symmetric across the perpendicular bisector of DC, thus $\angle EOD = \angle FOC = 60°$ and $\angle EOF = 120° - 60° = 60°$.



Let the radius of the circle be r, we first find the area enclosed by the chord and arc EF by considering the circular section EOF. We know $\triangle EOF$ is equilateral, which gives the area of section EOF:

$$\frac{\pi r^2}{6} - \frac{r(\frac{\sqrt{3}}{2}r)}{2} = r^2(\frac{\pi}{6} - \frac{\sqrt{3}}{4})$$

Similarly, we find the area enclosed by the chord and arc EC, using the section EOC. The area of $\triangle EOC$ is half of the area of $\triangle DEC$ as O is the midpoint of DC, which gives the area of section EOC:

$$\frac{\pi r^2}{3} - \frac{1}{2} \cdot \frac{r(r\sqrt{3})}{2} = r^2(\frac{\pi}{3} - \frac{\sqrt{3}}{4})$$

The shaded area is the difference between these two areas, hence our desired ratio is,

$$\frac{\text{Shaded}}{[DEC]} = \frac{r^2(\frac{\pi}{3} - \frac{\sqrt{3}}{4}) - r^2(\frac{\pi}{6} - \frac{\sqrt{3}}{4})}{r^2\frac{\sqrt{3}}{2}} = \boxed{\frac{\pi\sqrt{3}}{9}}$$

Solution 2 by Mr. Wei Wang: Let O be the center of the circumcircle and R be the length of the radius. Since $\triangle DEC$ is a right angled triangle, we know that the hypotenuse lies on the diameter of its circumcircle. By Central Angle Theorem (or Thales' Theorem) we know $\angle DOE = 60°$. By symmetry, we know $\angle FOC = 60°$ as well. We split the shaded region into two parts, namely the are bounded by FC and arc FC and $\triangle CFE$.

Since $\triangle DEC$ is a 30-60-90 triangle, $DE = \frac{1}{2} \cdot 2R = R$ and $EC = \frac{\sqrt{3}}{2} \cdot 2R = \sqrt{3}R$. Hence $[EOC] = \frac{R\cdot\sqrt{3}R}{2} = \frac{\sqrt{3}}{2} \cdot R^2$. From here we note that $\triangle DOE$ and $\triangle EOC$ have the same height, and both have a radii as a base. All radii are the same length, which implies that $[DOE] = [EOC] = \frac{1}{2}[DEC]$. Note that $EF||OC$. $\triangle FOC$ is an isosceles triangle we have $\angle FCO = \frac{180°-60°}{2} = 60°$, hence $EO||FC$. We have $\triangle EOC \cong CFE$ by SSS congruence (as EOCF is a parallelogram). Thus $[CFE] = \frac{1}{2}(\frac{\sqrt{3}}{2}R^2) = \frac{\sqrt{3}}{4}R^2$

From here we find the area enclosed by FC and the arc FC. We determined that $\angle FOC = \angle FCO = \angle OFC = 60°$, hence $\triangle FCO \cong \triangle DOE$ and the area between arc FC and chord FC is $\frac{1}{6}\pi R^2 - \frac{\sqrt{3}}{4}R^2$. Thus our desired ratio is

$$\frac{\text{Shaded}}{DEC} = \frac{R^2(\frac{\sqrt{3}}{4} + (\frac{1}{6}\pi - \frac{\sqrt{3}}{4}))}{R^2\frac{\sqrt{3}}{2}} = \boxed{\frac{\pi\sqrt{3}}{9}}$$

Problem 8

$10!$ can be expressed as $2^a \cdot 3^b \cdot 5^c \cdot \ell$ where ℓ is not divisible by 2, 3, or 5. Determine $a + b + c$.

Solution: Remember $n! = n(n-1)\ldots(2)(1)$. We will deal with each factor individually, we have

$$\begin{aligned}
10! &= 10 \cdot 9 \cdot 8 \ldots 2 \cdot 1 \\
&= (2 \cdot 4 \cdot 6 \cdot 8 \cdot 10)(3 \cdot 5 \cdot 7 \cdot 9) \\
&= 2^8(1 \cdot 1 \cdot 3 \cdot 1 \cdot 5)(3 \cdot 5 \cdot 7 \cdot 9) \\
&= 2^8 \cdot 3^4 \cdot 5 \cdot 5 \cdot 7 \\
&= 2^8 \cdot 3^4 \cdot 5^2 \cdot 7 \\
a &= 8, b = 4, c = 2 \\
a + b + c &= \boxed{14}
\end{aligned}$$

Problem 9

Levin Kin is competing in a math competition with 4 other contestants. Each contestant plays 3 games head-to-head with each other contestant, playing 12 games overall. The contestant with the most wins overall wins the entire tournament. How many games does Levin need to win in order to guarantee he will win the competition, without anyone else tying him for first?

Solution: As the rules states, winning a game causes the opposing contestant to lose. We instead consider the maximum number of times Levin Kin can lose. If Levin Kin loses a total of n times, where n is a positive integer, then he must cause all of the other contestants to lose more than n times. Since there are four contestants, and Levin wins a total of $12 - n$ times, we have $4 \cdot (n + 1) \leq 12 - n \Rightarrow 5n \leq 8$. Thus the maximum number of times Levin can lose is 1, which implies Levin wins $12 - 1 = \boxed{11}$ times.

Problem 10

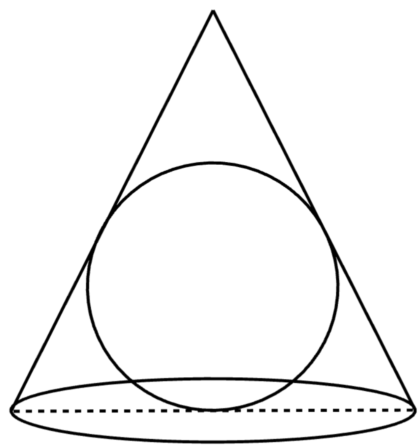
John the Zheng boat was rowing downstream at a speed of 17 km/h. After 35 minutes, he turned around and rowed back upstream the same distance. If his average speed was 10 km/h, find the speed at which he rowed upstream.

Solution: 35 minutes $= \frac{7}{12}$ hour. The distance he rowed downstream is $17 \cdot \frac{7}{12} = \frac{119}{12}$ km. Since he rows there and back, the total distance is double, which is $\frac{119}{6}$ km. Let the speed at which he rowed upstream be s. We know the total time is $\frac{7}{12} + \frac{\frac{119}{12}}{s}$. His average speed is 10 km/h, so solving for s gives

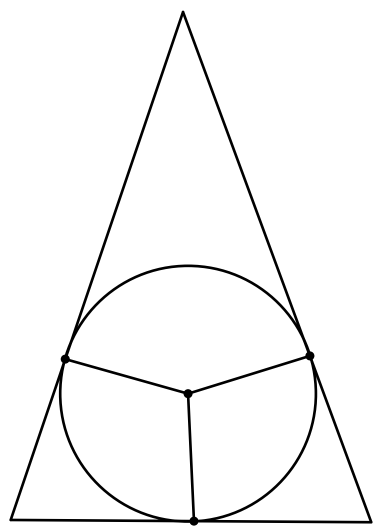
$$\frac{\frac{119}{6}}{\frac{7}{12} + \frac{\frac{119}{12}}{s}} = 10 \Rightarrow s = \boxed{\frac{85}{12} \text{ km/h}}$$

Problem 11

What is the volume of the sphere inscribed inside a cone with base radius 10 cm and height of 15 cm?



Solution: Take the cross section through the height of the cone. The resulting circle is the incenter of the isosceles triangle with base 10 cm and height 15 cm.



Let r be the radius of this circle (and therefore the sphere), if A, B, C are the vertices of the triangle, we know $r(\frac{AB+BC+AC}{2}) = [ABC]$.

$$r = \frac{2[ABC]}{AB+BC+AC} = \frac{2 \cdot \frac{(10)(15)}{2}}{10+5\sqrt{10}+5\sqrt{10}} = \frac{15}{1+\sqrt{10}} = \frac{15(1-\sqrt{10})}{-9} = \frac{5}{3}(\sqrt{10}-1)$$

From here we know the volume of a sphere is $\frac{4}{3}\pi r^3$, which results in

$$V = \frac{4}{3}\pi(\frac{5}{3}(\sqrt{10}-1))^3 = \boxed{\frac{500\pi}{81}(13\sqrt{10}-31) \approx 196}$$

Problem 12

Find the number integers x such that $\dfrac{x^2+3x-42}{x-3}$ is an integer.

Solution: We observe that we can divide out the polynomials as follows.

$$\frac{x^2+3x-42}{x-3} = \frac{((x-3)^2+6x-9)+3x-42}{x-3}$$
$$= \frac{((x-3)^2+9(x-3)+27-54}{x-3}$$
$$= x+9-\frac{27}{x-3}$$

Which is only an integer if $x-\frac{27}{x-3}$ is an integer. Let this integer be k.

$$x-\frac{27}{x-3} = k$$
$$x^2-3x-27 = kx-3k$$
$$x^2-(3+k)x-(27-3k) = 0$$

This only has integer solutions when the discriminant is a perfect square, hence for some integer r, we have $r^2 = (3+k)^2-4(-(27-3k))$. We can rearrange and use difference of squares.

$$r^2 = (3+k)^2-4(-(27-3k))$$
$$= k^2+6k-12k+9+108$$
$$= (k-3)^2+108$$
$$(r-k+3)(r+k-3) = 108$$

Each factor is an integer and the product is 108, thus and we can find the 8 divisor pairs (4 negative, 4 positive) that correspond to integer solutions in r, k.

Remark: The solutions are equivalent to stating that $x-3$ must divide 27. 27 has 4 divisors, each of which can be positive or negative, hence the number of integers x such that the given expression is an integer is $4 \cdot 2 = \boxed{8}$. The corresponding values of x are $\{\pm4, \pm6, \pm12, \pm30\}$.

Problem 13

How many integers from 1-1000 are spelt with 21 letters? (Ex. four hundred twenty four).

Solution: We consider the following list.

Word	Letters	Word	Letter
"one"	3	"eleven"	6
"two"	3	"twelve"	6
"three"	5	"thirteen"	8
"four"	4	"fourteen"	8
"five"	4	"fifteen"	7
"six"	3	"sixteen"	7
"seven"	5	"seventeen"	9
"eight"	5	"eighteen"	8
"nine"	4	"nineteen"	8

Word	Letters	Word	Letters
"ten"	3	"twenty"	6
"thirty"	6	"seventy"	7
"fourty"	6	"eighty"	6
"fifty"	5	"ninety"	6
"sixty"	5	"hundred"	7

Thus we note that any number must with 21 letters must be greater than one hundred. There are three spots to place a number. All numbers are of the form $\underset{Hundreds}{\underline{\hspace{2cm}}}$ hundred $\underset{Tens}{\underline{\hspace{1cm}}}$ $\underset{Ones}{\underline{\hspace{1cm}}}$. Where the Hundreds, Tens and Ones must sum to 14 letters.

Case 1: Tens and Ones combined, i.e. "thirteen".
The only possibility is 5+9=14, which has three possibilities - "three", "seven", or "eight" hundred "seventeen".

Case 2: Tens and Ones separate, i.e. "twenty seven".

2.1: Tens of length 5 ("fifty","sixty").
The length of the Ones and Hundreds are either of length 4, 5 or 5, 4 respectively. There are $3 \cdot 3 + 3 \cdot 3 = 18$ possibilities.

2.2: Tens of length 6
The length of the Ones and Hundreds are either of length 4, 4, or 3, 5, or 5, 3 respectively. There are $3 \cdot 3 + 3 \cdot 3 + 3 \cdot 3 = 27$ possibilities.

2.3: Tens of length 7
The length of the Ones and Hundreds are either of length 3, 4, or 4, 3 respectively. There are $3 \cdot 3 + 3 \cdot 3 = 18$ possibilities.

Combining the number of possibilities from all cases gives

$$(2) + (\cdot 18 + 5 \cdot 27 + 1 \cdot 18) = \boxed{192}$$

Problem 14

Determine two positive integers a, b such that $a^2 + b^2 = 10201$.

Solutions: By inspection, we find $(20, 99)$. It is important to see that

$$100^2 = 101^2 - 201 \to 99^2 = 100^2 - 199 \to 199 + 201 = 400 = 20^2$$

Final Proof

Problem 15

The integers from 1 to 100 are written on a whiteboard. Aaron circles 51 of these integers. Prove there must be a circled integer that is the multiple of another circled integer.

Solution: We represent every integer from 1 to 100 in the form $2^m n$, for some non-negative integer $m \leq 6$ and some odd positive integer $n < 100$. We will prove there exists two integers with the same value n, and therefore one must be a multiple of the other.

We group all the integers from 1 to 100; each integer with the same value n is put into one group. Since n can be any odd integer from 1 to 100, there are 50 possible values of n. Hence, there are 50 groups. However, Aaron must circle 51 integers, by the Pigeonhole Principle, he must choose at least two integers from one group (both of which have the same value n).

WLOG, let these two integers be $2^a n$ and $2^b n$ such that $a > b$ and a, b are non-negative integers and n is an odd positive integer. Thus $\frac{2^a n}{2^b n} = 2^{a-b}$, which is a positive integer. Therefore given any two integers with the same value n, one must be a multiple of another.

Marking Criteria:

1 - For any one (or all) of the following:

- Consider the unique case of 50 odd numbers and one even number.
- *Mention* a grouping of pigeons and holes to use pigeonhole principle on.
- Partitioning 100 integers in some way (not necessarily rigorously to prove the problem).

4 - Rigorous partition into odds and evens such that pigeonhole principle can be accurately used.

2 - Proof method always considers the case of even multiples of each other, i.e. 2, 4, 8 etc.

1 - Final mark awarded for complete proof with only trivial errors.

Note: An alternate solution with partitioning from 1-50 and 51-100 may be possible but is difficult to rigorously prove. For those who get significant progress on this method, award at most 3 marks.